$pq \neq p \cap q$ Example (M. Z.)

Another example

$y = x^2$

$\mathbb{Q}[x]$ $\quad p = (x)$

$p^2 = (x^2)$

$y = 0$

$$\mathbb{Q}[x,y]/(y-x^2, y) \cong \boxed{\mathbb{Q}[x]/(x^2)} \neq \mathbb{Q}[x]/(x)$$

R fixed ring

look at homomorphisms $R \to R'$ another ring

$\varphi: R \to k$

Ker φ is prime

if $a \subset R$ is prime $\Rightarrow$ R/a is a domain

$R \to R/a \subset F(R/a)$ is a field

Universal property:

$\forall$ field k' and $\varphi': R \to k'$ homomorphism s.t. $\operatorname{Ker}\varphi' = a$ $\exists!\, \psi$ which makes diagram commutative

$$\begin{tikzcd} R \arrow[r, "\varphi'"] \arrow[dr, "\varphi"'] & k' \\ & F(R/a) \arrow[u, dotted, "\psi"'] \end{tikzcd}$$

Misha Elkner

(Remember any homom. of fields is a field extension)

$\mathbb{Q}[X]/X^2$ is 1 point

(If $a \subset R$ is a radical ideal corresponding to $Z(a) \subset \operatorname{Spec} R$ then $Z(a) = \operatorname{Spec}(R/a)$ and R/a is called the coordinate ring) Moritz

$I(Z(a)) = \operatorname{rad}(a)$

Chiara M.

Conclusion a is prime then we have $Z(a) \subset \operatorname{Spec}(R)$ and $a \in \operatorname{Spec}(R)$ (because a is a point)

Today: Nullstellensatz

Connection between ideals and actual sets of solutions

contrary to subsets of Spec R.

Setup: Fix a field k, consider k-algebra R. (**Def** R is a k-algebra if we prescribe a map $k \to R$ (necessarily injective)

Example $R = k[X_1, \dots, X_n]/(f_1, \dots, f_m)$)

A homomorphism of k-algebras is a ring homomorphism $R \to R'$ s.t.

this diagram is commutative:
$$\begin{tikzcd} & R \arrow[r] & R' \\ k \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

(don't want $\mathbb{C}[X] \to \mathbb{C}[X]$, $a_i X^i \mapsto \bar{a}_i X^i$).

Prop (Tayebeh) a k-algebra is a vector space over k (not necc. fin. dim)

$\to$ Category of k-algebras.

Many constructions for rings also apply to algebras, e.g. Functor of points.

Fix R k-algebra

Various kinds of points:

1) Spec R each $p \in \operatorname{Spec} R$ corresponds to $R \to F(R/p)$

$F(R/p)$ is a field extension of k (may be infinite)

2) Maximal ideals

$\operatorname{Specm}(R) \subset \operatorname{Spec}(R)$ corresponds to $R \to \underbrace{R/p}_{\text{field (extension of } k)}$

3) Geometric points: Let $\bar{k}$ be the algebraic closure of k (e.g. $\bar{k} = k$ if k is alg. closed)

algebra homomorphisms $R \to \bar{k}$.

(actually correspond to solutions of systems of equations when $R = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$.

Note: $\bar{k} = \{$roots of all polynomials in one variable with k-coefficients$\}$.

any $\varphi: R \to \bar{k}$ has image contained in some finite extension of k.

Prop $\varphi(R)$ is a field $\Rightarrow$ $\operatorname{Ker} \varphi$ is maximal.

Thm if $m \subset R$ is maximal, R finitely generated k-algebra, $\Rightarrow$ R/m is an algebraic extension of k.

Cor If $k = \bar{k}$ then maximal ideals are in bijection with k-points.

(ideals $\longleftrightarrow$ subsets of Specm(R) will be clear).

$\updownarrow$ $\leftarrow$ Nullstellensatz.

geometric points

break $\to$ 10:43

Proofs

Prop $k \subset L$ finite field extension $L \supset R \supset k$

R ring, then R is a field.

Proof $x \in R$ Let $x^n + a_{n-1}x^{n-1} + \dots + a_0$ be the minimal polynomial of x $\Rightarrow a_0 \neq 0 \Rightarrow x^{-1} = \dfrac{-x^{n-1} - a_{n-1}x^{n-2} - \dots - a_1}{a_0}$.

$\Rightarrow x^{-1} \in R$. ∎

(geometric points correspond to maximal ideals).

Proof of Thm: Let m be a maximal ideal

Consider R/m. This is a field extension of k.

R/m is a finitely generated algebra over k.

We want to prove: R/m is a finite extension. (fin. dim $/k$).

Let $x_1, \dots, x_n$ be generators of R/m as alg. $/k$.

$k \subset k(x_1) \subset k(x_1, x_2) \subset \dots \subset R/m$

Let $k(x_1, \dots, x_i)$ be the subfield of R/m generated by k, and $x_1, \dots, x_i$.

Go from right to left.

$K_i = k(x_1, \dots, x_i) \subset K_{i+1}$

$K_{i+1} = K_i(x_{i+1})$.

We know $K_n = R/m$ is finitely generated algebra $/k$.

Look at K_{n-1}

2 cases: 1) K_n/K_{n-1} finite field extension

2) infinite.

I case 1)

Claim: K_{n-1} is also finitely generated alg $/k$.

Consider embedding $\varphi: K_n \to$ matrices over K_{n-1} of size $r \times r$. ($r = \dim K_n/K_{n-1}$).

Let $y_1, \dots, y_m$ be the algebra generators of K_n/k.

Take $\varphi(y_1), \dots, \varphi(y_m)$, take all their entries

Obtain $m \cdot r^2$ elements of K_{n-1}. These generate K_{n-1}:

Pick $z \in K_{n-1}$, z is a polynomial of $y_1, \dots, y_m$ with coeffs in k. ($z \in K_n$) $\Rightarrow \varphi(z)$ as a matrix has entries polynomials in these $m \cdot r^2$ elements.

but since $z \in K_{n-1}$, $\varphi(z) = \begin{pmatrix} z & & 0 \\ & z & \\ 0 & & z \end{pmatrix}$, so in particular z is a polynomial in these $m \cdot r^2$ elements.

Continue for $n-1$, $n-2$:

K_n is finite $/K_{n-1}$ $\qquad$ K_{n-1} is f. gen $/k$

K_{n-1} is finite $/K_{n-2}$ $\qquad$ K_{n-2} is f. gen $/k$.

$\vdots$

until

K_i is infinite $/K_{i-1}$. (Case 2)

hence $K_i = K_{i-1}(x)$ (field of rational functions)

also K_i is finitely generated algebra $/k$, hence over K_{i-1}.

Suppose K_i is generated by $\dfrac{p_1(x)}{q_1(x)}, \dots \dfrac{p_m(x)}{q_m(x)}$.

$\Rightarrow$ every rational function must have denominator = product of powers of $q_1, \dots, q_m$.

On the other hand consider $\dfrac{1}{1 + q_1(x)q_2(x) \dots q_m(x)}$, has denominator relatively prime to $q_1, \dots, q_m$ ⚡.

Conclusion: Case 2 doesn't happen, so $\forall i$ K_i/K_{i-1} is finite. $\Rightarrow$ $R/m = K_n/k$ is finite.

∎

Map

geometric pts $\rightarrow$ maximal ideals

several to one

$\varphi_1 : R \to \bar{k}$

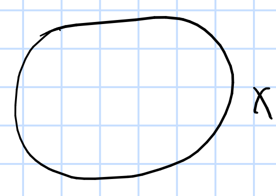
$\varphi_2 : R \to \bar{k}$ $\Rightarrow$ $R/m \rightrightarrows \bar{k}$

$m = \ker \varphi_1 = \ker \varphi_2$

R/m finite $/k$

different geo. pts correspond to different embeddings $R/m \to \bar{k}$.

Examples



$x^2+y^2=1$ $/\mathbb{R}$

$\bar{\mathbb{R}} = \mathbb{C}$

for $(x,y) \in \mathbb{R}$ $x^2+y^2=1$

1 geometric point.

$(R/m = k)$.

φ_1 $x = i$ $y = \sqrt{2}$

φ_2 $x = -i$ $y = \sqrt{2}$ 2 geometric points.

correspond to the same max. ideal

because

$\varphi_1 : R \to \mathbb{C}$

$\varphi_2 : R \to \mathbb{C}$

$R = \mathbb{R}[x,y]/(x^2+y^2-1)$

$\mathbb{R}$ reals

$$\begin{array}{ccc} R & \xrightarrow{\varphi_1} & \mathbb{C} \\ & \searrow_{\varphi_2} & \uparrow \text{complex conjugation} \\ & & \mathbb{C} \end{array}$$

$\Rightarrow \ker \varphi_1 = \ker \varphi_2$

Similarly in general.

Consider $Z(a) \subset \mathrm{Spec}(R)$

Let $Z_m(a) \subset \mathrm{Specm}(R)$ $Z_m(a) = Z(a) \cap \mathrm{Specm}(R)$.

Claim $I(Z_m(a)) = \mathrm{rad}(a)$. ($R$ finitely gen over k).

Pf (similarly $I Z(a) = \mathrm{rad}(a)$).

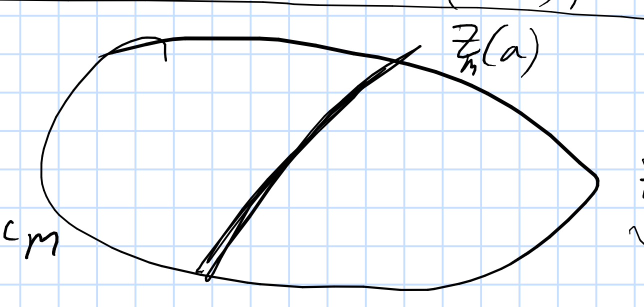
Consider $a \subset I(Z_m(a))$

Suppose $f \in I Z_m(a)$.

Consider $R/a[x]/(fx=1)$ if this ring is $\neq 0$

then $\mathrm{Specm}(R/a[x]/(fx=1)) \neq \emptyset$. $\Rightarrow \exists p \in \mathrm{Specm}(R)$ s.t. $f(p) \neq 0$ $p \in Z_m(a)$ $\times$.

Picture



Spec_m $Z_m(a)$

$Z_m(a) \cap \{f \neq 0\} = \emptyset$

$\Rightarrow R/a[x]/(fx-1) = 0$

$\Rightarrow 1 = 0$ in $R/a[x]$ mod $(fx-1)$

$\Rightarrow (fx-1)(c_n x^n + \dots + c_0) = 1$ $c_i f = c_{i+1}$

$\Rightarrow c_0 = 1$ $c_1 = f$ .. $c_n = f^n$ $c_{n+1} = f^{n+1} = 0$

$\Rightarrow f \in \mathrm{rad}(a)$.

Cor $\mathrm{rad}(a) = \bigcap_{\substack{m \text{ maximal} \\ m \supset a}} m$ (if R is finitely generated $/k$).

More geometry.

Zariski topology made Spec R into a top space.

$\varphi : R \to R'$ induces $\varphi^* : \mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ is continuous.

because closed sets = zero sets

$(\varphi^*)^{-1} Z(f_1, \dots, f_n) = Z(\varphi(f_1), \dots, \varphi(f_n))$.

Far from true that all continuous maps come from homomorphisms.

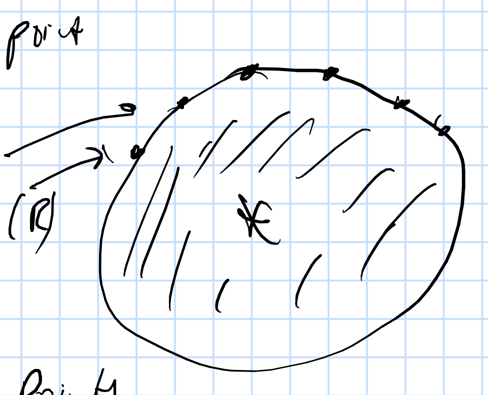
Ex $R = k[x]$ $\mathrm{Spec}(R) = $ . . . . $\}$ $\mathrm{Specm}(R)$

$*$ (0)

What is the topology? Closed sets are finite subsets of $\mathrm{Specm}(R)$ and the whole $\mathrm{Spec}(R)$.

any bijection $\mathrm{Specm}(R) \to \mathrm{Specm}(R)$ induces a continuous map.

Notice $\overline{*} = \mathrm{Spec}(R)$. $*$ is called generic point



$\mathrm{Specm}(R)$ $*$

Non hausdorff "bad space".

Points

Prop $p \in \mathrm{Spec}(R)$ so that $\{p\}$ is closed are precisely the maximal ideals.

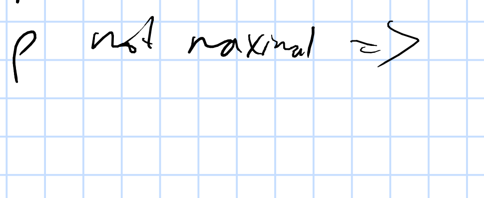
Proof $\overline{\{p\}} = Z(I(\{p\})) = Z(p) = \{q \in \mathrm{Spec}(R) \mid p \subset q\}$

p maximal $\Rightarrow \overline{\{p\}} = \{p\}$

p not maximal $\Rightarrow$ $\exists q$ maximal $q \supset p$

$\overline{\{p\}} \ni q$ $\Rightarrow \{p\}$ is not closed. ∎

In more dimensions



prime ideal

p generic point. "moves along $Z_m(p)$."

Roughly, $\overline{\{*\}} = \{$smaller points contained in $Z_m(p)\}$

Def A scheme is a ringed space locally isomorphic to $\mathrm{Spec}(R)$ for some rings R.

Def A ringed space is a top. space X together with a "structure sheaf":

$\forall\, U \subset X$ open a ring $\mathcal{O}_U$, for each

$U \subset V$ open a restriction homomorphism $\mathcal{O}_V \to \mathcal{O}_U$

Such that

1) if $\{U_\lambda\}_{\lambda \in \Lambda}$ so that $U = \bigcup_\lambda U_\lambda$ then

$\mathcal{O}_U \to \prod_\lambda \mathcal{O}_{U_\lambda}$ is injective

2) in the above situation if $f_\lambda \in \mathcal{O}_{U_\lambda}$ is given $\forall \lambda$ s.t.

$\forall \lambda, \mu$ $f_\lambda|_{U_\lambda \cap U_\mu} = f_\mu|_{U_\lambda \cap U_\mu}$ in $\mathcal{O}_{U_\lambda \cap U_\mu}$

then $\exists f \in \mathcal{O}_U$ whose restriction to U_λ is f_λ.

0) $U \subset V \subset W$: $\mathcal{O}_W \to \mathcal{O}_V \to \mathcal{O}_U = \mathcal{O}_W \to \mathcal{O}_U$.