R ring

$f \in R$ functions

p prime $\subset R$ points

Prop If U_f affine open is covered by U_{g_i} affine opens then 1) $\mathcal{O}(U_f) \to \prod \mathcal{O}(U_{g_i})$ is injective

2) If $x_i \in \mathcal{O}(U_{g_i})$ satisfy

$$\forall i,j \quad \rho_{U_{g_i} \cap U_{g_j}}(x_i) = \rho_{U_{g_i} \cap U_{g_j}}(x_j)$$

$$\Rightarrow \exists\ x \in \mathcal{O}(U_f) \text{ s.t. } \rho_{U_{g_i}}(x) = x_i$$

This is called the sheaf property.

Proof Last time:

$f^n = \sum a_i g_i$ some $a_i \in R$, $a_i = 0$ except finitely many.

injectivity: x suppose $\rho_{U_{g_i}}(x) = 0$

$\Rightarrow$ in the ring $R[t]/(tg_i - 1)$

x is zero $\Leftrightarrow$

$$x = p(t)(tg_i - 1) =$$
$$= (c_k t^k + \dots + c_0)(tg_i - 1)$$
$$= c_k g_i t^{k+1} + (c_{k-1} g_i - c_k) t^k + \dots - c_0$$

$c_0 = -x$, $c_1 = -g_i x$ ... $c_k = -g_i^k x$

$c_k = 0$

so x goes to 0 in $\mathcal{O}(U_g) \Leftrightarrow$

$g^k x = 0$, some k.

So assume $g_i^k x = 0$ $\forall i$ which enter the expansion $f^n = \sum a_i g_i$ (x)

take f^{nkm}

$m = \#$ of non-zero coefficients in (x)

in expansion of f^{nkm} each term contains at least one $g_i^{\geq k}$ $\Rightarrow$

$f^{nkm} x = 0 \Rightarrow x = 0.$

Part II. Suppose we know $\rho_{U_i}(x) = x_i$, we want to construct x.

replace the covering $\{U_i\}$ by the sub covering containing only U_i for which $a_i \neq 0$. This is a finite covering. $U_1, \dots, U_m$ (see ex.)

and we have $x_i \in \mathcal{O}(U_i)$ $U_i = U_{g_i}$

therefore $x_i = \frac{y_i}{g_i^k}$ ("$=$" x)

look at $x f^{knm} =$ big sum

each term some i will contain g_i^k, replace $x g_i^k$ by y_i.

Again, write $f^{kmn} = \sum g_i^k h_i$,

Let $x = \frac{\sum y_i h_i}{f^{kmn}}$.

Claim that x "works"

we need to show that

$\rho_{U_i}(x) = x_i \Leftrightarrow$

$\rho_{U_i}(x f^{kmn}) = x_i f^{kmn}$

$\Leftrightarrow \boxed{\rho_{U_i}\left(\sum y_j h_j\right) = x_i \sum g_j^k h_j}$

think $\Big($ $y_j = x_j g_j^k$

$\sum x_j g_j^k h_j = x_i \sum g_j^k h_j$

$x_j g_j^k = x_i g_j^k$ $\Big)$

problem: x_j is not in $\mathcal{O}(U_i)$.

Let's restrict to $U_i \cap U_j$.

we have $x_j g_j^k =$

$y_j h_j = x_i g_j^k h_j$

On $U_i \cap U_j$ this holds ($x_i = x_j$)

We need: $y_j = x_i g_j^k$

We know: $x_i = x_j$ on $U_i \cap U_j$

$$\Leftrightarrow \left(\frac{y_i}{g_i^k} - \frac{y_j}{g_j^k}\right)(g_i g_j)^{m'} = 0$$

$$\Rightarrow (y_j - x_i g_j^k)(g_i g_j)^{m'} = 0$$

We want on U_i:

$\sum y_j h_j = x_i \sum g_j^k h_j$ Homework.

We have $\boxed{(y_j - x_i g_j^k)(g_i g_j)^{m'} = 0.}$

$\Downarrow$

$\boxed{\begin{array}{l}(y_j - x_i g_j^k)\, g_j^{m'} = 0 \text{ on } U_i \\ y_j g_j^{m'} - x_i g_j^{k+m'} = 0\end{array}}$

Before the proof: Replace y_j by $y_j g_j^{m'}$

k by $k+m'$

hence we can assume

$y_j - x_i g_j^k = 0$ in $\mathcal{O}(U_i)$ $\forall i,j$.

Continue the proof from the beginning

Continue on dimensions.

Recall $p \subset R$ prime: $ht(p)$ = maximal length of a chain $p_0 \subset \dots \subset p_n = p$

Another idea

Def $g(p) = \min_k \{ p \text{ is a minimal prime containing } \underbrace{(f_1, \dots, f_k)}_{\text{not always radical}} \}$.

its radical not always prime, $\exists$ finitely many minimal primes containing it.

Theorem R Noetherian $\Rightarrow$ $g(p) = ht(p)$ $\forall$ primes p.

Suppose $p_0 \subseteq p_1 \subseteq \dots \subseteq p_k$ is a maximal chain

then p_{i+1} is a minimal prime containing $p_i + (f_i)$, some $f_i \subset p_{i+1}$

$\Rightarrow$ p_k is a minimal prime containing $(f_1, \dots, f_k)$

$\Rightarrow$ $g(p) \leq ht(p)$. EASY!

We want: $ht(p) \leq g(p)$

in particular, $ht(p) \neq \infty$.

Equivalently if p is a minimal prime containing $(f_1, \dots, f_k)$ $\Rightarrow$ $ht(p) \leq k$.

Examples $k=0$: clear

$k=1$: Krull's Hauptidealsatz

Idea: induction on k.

Let's prove the step:

p minimal prime $\supset (f_1, \dots, f_k)$

Suppose $q \subset p$ prime so that $\nexists$ no primes in between.

W.L.O.G. $f_1 \notin q$ $\Rightarrow$

p is a minimal prime containing $q + (f_1)$

We want to replace $f_2, \dots, f_k$ by some elements $g_2, \dots, g_k \in q$ so that p is a minimal prime containing $(f_1, g_2, \dots, g_k)$

If we succeed, then

$$(g_2, \dots, g_k) \subset \underset{\text{minimal prime}}{q'} \subset q \subset p$$

minimal containing principal (f_1)

in $R/(g_2, \dots, g_k)$ $p \supset$ a minimal prime containing (f_1)

q is another prime, by case $k=1$ (Krull's Hauptidealsatz) we have q is a minimal prime containing $(g_2, \dots, g_k)$ $\Rightarrow$ $ht(q) \leq k-1$

q arbitrary $\Rightarrow$ $ht(p) \leq k$.

$f_1, \dots, f_k$

$f_2, \dots, f_k \to g_2, \dots, g_k \in q$.

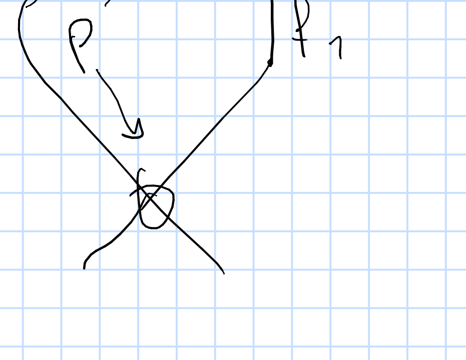
$rad(q + (f_1))$ has a minimal prime p

$(f_1, \dots, f_k) \subset p$ $\qquad f_i \in rad(q + (f_1))$?

Obstruction: maybe there are many minimal primes.

Solution: localization at p.

$rad(f_1, \dots, f_k) = p_1 \cap \dots \cap p_i$



Localization

Setup R ring

$S \subset R$ multiplicative set if $1 \in S$, $x, y \in S \Rightarrow xy \in S$.

Example ideal $p \subset R$ is prime $\Leftrightarrow$ $R \setminus p$ is multiplicative.

Form localization $S^{-1}R$ $\left(= R_p \text{ if } S = R \setminus p\right)$

as a set $S^{-1}R = \left\{ \frac{x}{s} \mid s \in S, x \in R \right\} / \sim$

$\sim : \frac{x}{s} = \frac{x'}{s'}$ if $\exists u \in S$ s.t. $(xs' - sx')u = 0$.

$\frac{x}{s} = 0 \Leftrightarrow \exists u \in S$ s.t. $xu = 0$.

(Recall for $O(U_f)$ we have seen that $x \in R$ goes to 0 in $O(U_f)$ if $xf^n = 0$ some n. So this was a special case for $S = \{f^n \mid n \geq 0\}$)

As a set $S^{-1}R$ is clear

Any fraction $\frac{x}{s}$ $\forall u \in S$ we have $\frac{xu}{su} = \frac{x}{s}$ $(xus - xus = 0)$

Equiv. relation: $\frac{x}{s} = \frac{x'}{s'}$ $\quad \frac{x'}{s'} = \frac{x''}{s''}$

$\Downarrow$

$(xs' - x's)u = 0$

$$\frac{x}{s} = \frac{xs'u}{ss'u} = \frac{x'su}{ss'u} = \frac{x'}{s'}$$

$\Rightarrow \frac{x}{s} = \frac{x'}{s'} \Leftrightarrow \exists u_1, u_2$. s.t.

$$\left.\begin{aligned} xu_1 &= x'u_2 \\ su_1 &= s'u_2 \\ x'u_3 &= x''u_4 \\ s'u_3 &= s''u_4 \end{aligned}\right\} \quad \begin{aligned} xu_1u_3 &= x'u_2u_3 = x''u_2u_4 \\ s u_1u_3 &= s'u_2u_3 = s''u_2u_4 \end{aligned}$$

$\frac{x'}{s'} = \frac{x''}{s''}$

$\Downarrow$

$\frac{x}{s} = \frac{x''}{s''}$.

Abelian group: Any 2 elements can be made to have the same denominator, and $\frac{x}{s} + \frac{x'}{s} = \frac{x+x'}{s}$.

Product: $\frac{x}{s}\frac{x'}{s'} = \frac{xx'}{ss'}$.

$\frac{1}{1} = 1$ the unit.

So $S^{-1}R$ is a ring.

Universal property:

$\mathrm{Hom}(S^{-1}R, R') = \{ f : R \to R' \mid \forall s \in S \ f(s) \text{ is invertible} \}$

discussion:

$F_{S^{-1}R}(R') \subset F_R(R')$

$\Downarrow$

exists $R \to S^{-1}R$.

Construction of $R \to S^{-1}R$

$x \in R \to \frac{x}{1}$. Clearly a ring homomorphism.

__Proof__ of the universal property:

$\rightarrow$ Given $S^{-1}R \to R'$, compose it with $R \to S^{-1}R$, obtain $R \to R'$

$\forall s \in S$ it goes to $\frac{s}{1}$ in $S^{-1}R$, where it is invertible $\left(\frac{1}{s} \cdot \frac{s}{1} = 1\right)$

so in R' the image is also invertible

$\leftarrow$

$f : R \to R'$ s.t. $f(s)$ is invertible

we need $S^{-1}R \to R'$.

send $\frac{x}{s}$ to $f(x) \cdot \underbrace{f(s)^{-1}}_{\text{unique inverse}}$

if replace $\frac{x}{s}$ by $\frac{xu}{su}$ $u \in S$,

obtain $f(xu) f(su)^{-1} = f(x) f(u) f(u)^{-1} f(s)^{-1}$
$= f(x) f(s)^{-1}$.

So the map is well-defined.

abelian group homomorphism $\frac{x}{s} + \frac{x'}{s} \to \frac{f(x+x')}{f(s)}$ holds.

multiplicative also clear. $\lrcorner$

Ex R domain $R_{(0)} = S^{-1}R$
(0) prime $(S = R \setminus \{0\})$
$F(R) = R_{(0)}$.

__Ideals__ Given $a \subset R$ ideal we have $S^{-1}a = \left\{ \frac{x}{s} \in S^{-1}R \mid xu \in a \ (\text{some } u \in S) \right\}$

$xu \in a \Rightarrow xu'u \in a$

$\frac{x}{s} \in S^{-1}a \Leftrightarrow \frac{xu}{su} \in S^{-1}a$ according to our definition, so $S^{-1}a$ is well-defined.

w.r.t. $\pi : R \to S^{-1}R$

$S^{-1}a = \pi(a) \cdot S^{-1}R$ because

clearly $\pi(a) \subset S^{-1}a$, conversely

$\frac{x}{s}$ satisfies $xu \in a \Rightarrow \frac{x}{s} = \frac{xu}{su} =$
$= \frac{1}{su} \cdot \pi(xu)$
$\in \pi(a) S^{-1}R$.

given $a \subset S^{-1}R$ obtain $\pi^{-1}a \subset R$.

notice: $S^{-1}\pi^{-1}(a) = a$

$\quad \shortparallel$

$\pi(\pi^{-1}(a)) S^{-1}R \subset a$

Conversely $\frac{x}{s} \in a \Rightarrow \frac{x}{1} \in a$

$\Rightarrow x \in \pi^{-1}(a) \Rightarrow \frac{x}{s} = \frac{1}{s} \cdot \pi(x) \in \pi(\pi^{-1}(a)) S^{-1}R$.

__Example__ $S^{-1}(0) = (0)$

$\pi^{-1}(0) = \{ x \in R \mid xs = 0 \ (\text{some } s \in S) \}$

ideals of $S^{-1}R \xleftrightarrow[\text{via } \pi^{-1}]{\text{bijection}}$ subset of ideals of R

Hence R noetherian $\Rightarrow$ $S^{-1}R$ noetherian

__Remark__ if S is generated by finitely many elements $\Rightarrow S^{-1}R$ is finitely generated over R.
$\Rightarrow S^{-1}R$ is noetherian.

__Prime ideals__

$\pi^{-1}(\text{prime})$ is a prime, so $\mathrm{Spec}(S^{-1}R) \subset \mathrm{Spec}(R)$.

__Prop__ $p \subset R$ prime belongs to $\pi^{*}\mathrm{Spec}(S^{-1}R) \Leftrightarrow p \cap S = \emptyset$.

__Proof__ k field

A homomorphism $R \to k$ comes from a homo. $S^{-1}R \to k \Leftrightarrow$ $\forall s \in S$ goes to $\neq 0$ in k.
$\Leftrightarrow S \cap \mathrm{Ker}(R \to k) = \emptyset$. $\lrcorner$

__Prop__ If p is a prime $\subset R$

$\Rightarrow \mathrm{Spec}\, R_p \subset \mathrm{Spec}\, R$ consists of prime ideals contained in p.

$R_p = S^{-1}R$, $S = R \setminus p$

__Notation__: $a_p = S^{-1}a$

__Cor__ R_p has a unique maximal ideal $S^{-1}p = p_p$.

__Def__ A ring with a unique maximal ideal is called a local ring.

Before

$(f_1, \ldots, f_n) \subset p$ minimal

$q \subset p$ $\qquad q + (f_1) \subset p$ minimal

pass to R_p, think $R = R_p$

in R_p p becomes maximal

$\mathrm{rad}(f_1, \ldots, f_n) = p$

$\mathrm{rad}(q + (f_1)) = p \ni f_1, \ldots, f_n$

$\Rightarrow f_i^{n_i} = g_i + c_i f_1$ $\quad g_i \in q$, $c_i \in R$

$\Rightarrow \mathrm{rad}(g_1, \ldots, g_{n-1}, f_1) = p$.

we use the strategy we outlined.

Next time: Krull's Hauptidealsatz.