Examples from the exercises.

From Hartshorne;

k alg. closed field

$\mathbb{A}^3 = \mathrm{Specm}(k[x,y,z]) = k^3$

Let $Y = \{(t^3,t^4,t^5) \mid t \in k\} \subset \mathbb{A}^3$

$I = I(Y)$

1) Y irreducible

Prop if $f: X \to Y$ is a cont. map, X is irreducible $\Rightarrow$ $f(X)$ is irreducible. **Pf** Otherwise cover $f(X)$ by proper closed Z_1, Z_2 $\Rightarrow$ $f^{-1}(Z_1) \cup f^{-1}(Z_2) = X$

$Z_1 \neq \emptyset \Rightarrow f^{-1}(Z_1) \neq \emptyset$ $\quad (Z_1 \subset f(X))$

$x \notin Z_1 \Rightarrow f^{-1}(x) \neq \emptyset \quad f^{-1}(x) \cap Z_1 = \emptyset$.

So Z_1 is proper, Z_2 is also proper analogously.

($\mathbb{A}^1 \to \mathbb{A}^3$ $\quad t \to (t^3,t^4,t^5)$ is continuous (corresponds to a ring homomorphism $k[x,y,z] \to k[t]$ sending $x \to t^3$ $\quad y \to t^4, z \to t^5$. So continuous).

2) Find $\dim(Y)$.

we have $t \to (t^3,t^4,t^5)$ injective:

if (x,y,z) is in the image, then

$x = 0 \Rightarrow t = 0$

$x \neq 0 \Rightarrow t = y/x$.

So we have a partial inverse.

So Y is in bijection with $\mathbb{A}^1$. topology on $\mathbb{A}^1$ is cofinite, so the topology on Y is also cofinite.

$\Rightarrow \dim Y = 1$

By the way, the argument (x,y,z) in the image $\Rightarrow t = 0$ or $t = y/x$ works for arbitrary field.

and so the map on $\mathrm{Spec}(\mathbb{A}^1) \to \mathrm{Spec}(\mathbb{A}^3)$ is injective.

3) We have $\dim Y = 1$, so we can guess $ht(I(Y)) = 2$.

So $I(Y)$ is a minimal prime containing an ideal generated by 2 elements.

$f \in I(Y) \Leftrightarrow \forall t \in k \quad f(t^3,t^4,t^5) = 0$

$\Leftrightarrow \quad f(t^3,t^4,t^5) \in k[t]$ is zero.

any monomial in x,y,z, after the substitution becomes t^k, we say k is the degree of the monomial in x,y,z. Let's list all monomials in increasing degree

monomial	degree	substitution	ker
1		1	
x	3	t^3	
y	4	t^4	
z	5	t^5	
x^2	6	t^6	
xy	7	t^7	
xz, y^2	8	t^8, t^8	$xz-y^2$
x^3, yz	9	t^9, t^9	x^3-yz
x^2y, z^2	10		x^2y-z^2
$\vdots$			

Observe that the new elements in the kernel are combinations of

$\underset{a}{xz-y^2}, \quad \underset{b}{x^3-yz}, \quad \underset{c}{x^2y-z^2}$.

Conjecture: $I(Y) = (a,b,c)$.

pf Use these relations to decrease degree in y, z, so we obtain any $p \in k[x,y,z]$ can be written as

$p = a \cdot p_a + b \cdot p_b + c \cdot p_c + \alpha(x) + \beta(x)y + \gamma(x)z$.

$p_a, p_b, p_c \in k[x,y,z] \qquad \alpha, \beta, \gamma \in k[x]$.

$p(t^3,t^4,t^5) = 0 \Leftrightarrow \alpha(t^3) + \beta(t^3)t^4 + \gamma(t^3)t^5 = 0$

degrees $\equiv 0\,(3)$ $\quad$ degrees $\equiv 1\,(3)$ $\quad$ degrees $\equiv 2\,(3)$.

$p(t^3,t^4,t^5) = 0 \Leftrightarrow \alpha = \beta = \gamma = 0$.

So $I(Y) = (a,b,c)$.

3b) $I(Y)$ cannot be generated by 2 elements.

Consider a map $\varphi: I(Y) \to k^3$ given by sending any $p \in I(Y)$ to the coefficients of y^2, yz, z^2.

Notice: $\forall p \in I(Y) \quad \varphi(xp) = 0$

$\varphi(yp) = 0 \quad \varphi(zp) = 0$.

So if $I(Y)$ is generated by 2 elements say q_1, q_2, then

$\varphi(I(Y)) = k\varphi(q_1) + k\varphi(q_2) \Rightarrow$

$\dim \varphi(I(Y)) \leq 2$ contradiction.

(φ is clearly surjective)

4) Show that $\exists\, J \subset k[x,y,z]$ generated by 2 elements s.t. I is among the set of minimal primes containing J.

(I is prime by (1)).

It shouldn't matter how to choose 2 generators, take

$(a,b) = (y^2-xz, yz-x^3) = J$.

Consider $k[x,y,z]/J$.

Look at solutions to a, b over arbitrary field K.

$\begin{cases} y^2 = xz \\ yz = x^3 \end{cases}$

$\to$ $x = 0 \Rightarrow y = 0$, z arbitrary $\quad$ (I)

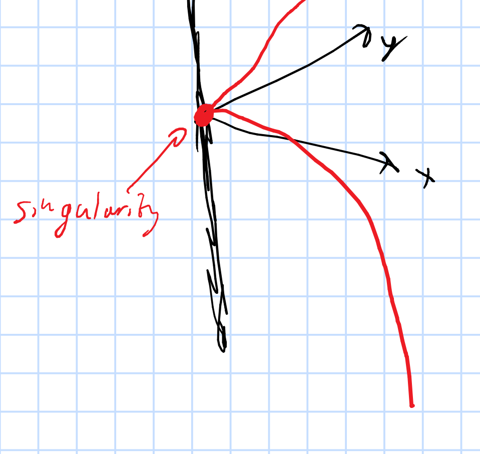
$\to$ $x \neq 0 \Rightarrow z = \dfrac{y^2}{x}$ $\quad$ (II)

$\dfrac{y^3}{x} = x^3 \Rightarrow y^3 = x^4$

$z^2 = \dfrac{y^4}{x^2} = yx^2$.

So $Z(J) = U \cup Y$ where $U = Z(x,y)$, so U, Y are the irreducible components of $Z(J)$.

$\Rightarrow$ minimal primes containing J are I and (x,y).



Schemes

Terminology

We work with ringed spaces, i.e. pair $(X, \mathcal{O}_X)$

top. space — "sheaf of functions" structure sheaf.

Locality **Def** Let P be some property of ringed spaces. We say that X is locally P if any of the 2 equivalent conditions holds:

1) $\exists$ open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of X such that U_λ is P for all λ

2) $\forall x \in X$ $\exists$ open nbh U_x such that U_x is P.

Examples A diff. manifold is locally $\underbrace{\text{isomorphic to } \mathbb{R}^n}_{P}$

Def X is an affine scheme if X is isomorphic to $\mathrm{Spec}(R)$ for a ring R. If k is a field, X is an affine scheme over k if X is isomorphic to $\mathrm{Spec}(R)$ for a k-algebra R.

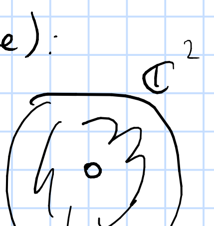
Def X is a scheme is X is locally an affine scheme.

Example Let X be an affine scheme, $U \subset X$ open subset. Define $\mathcal{O}_U$ by $\mathcal{O}_U(V) = \mathcal{O}_X(V)$ ($V \subset U$ open). Then $(U, \mathcal{O}_U)$ is a scheme.

Pf affine open sets (also called basic open sets or principal open sets) form a basis of topology $\Rightarrow$ $(U, \mathcal{O}_U)$ can be covered by affine schemes.

Example Scheme (not affine):

$U = \mathbb{C}^2 \setminus \{(0,0)\}$.



Claim: U is not affine.

Pf if U is affine, then $U = \mathrm{Spec}(\mathcal{O}(U))$, so let's compute $\mathcal{O}(U)$.

To compute $\mathcal{O}(U)$ cover U by basic open sets $\boxed{U_1 = \{x \neq 0\}}$, $U_2 = \{y \neq 0\}$.

more pedantic:

$U_1 = \{p \in \mathrm{Spec}\, \mathbb{C}[x,y] \mid x \notin p\}$

$U_2 = \ldots$

$U_1 = \mathrm{Spec}\, x^{-1}R$ $\quad U_2 = \mathrm{Spec}\, y^{-1}R$

$R = \mathbb{C}[x,y]$.

By definition $\mathcal{O}(U) = \left\{ \begin{array}{l} f \in \mathcal{O}(U_1) \\ g \in \mathcal{O}(U_2) \end{array} \middle| \; f|_{U_1 \cap U_2} = g|_{U_1 \cap U_2} \right\}$

$= \left\{ \begin{array}{l} f \in x^{-1}R \\ g \in y^{-1}R \end{array} \middle| \; f, g \text{ go to the same function in } (xy)^{-1}R \right\}$

$f = \dfrac{P(x,y)}{x^n} \quad g = \dfrac{Q(x,y)}{y^n}$

$\boxed{(xy)^{-1}R \to F(R)}$ is injective

$\dfrac{P(x,y)}{x^n} = \dfrac{Q(x,y)}{y^n} \Rightarrow P(x,y)\, y^n = Q(x,y)\, x^n$

$\Rightarrow x^n \mid P \quad y^n \mid Q$

$\Rightarrow f \in R,\ g \in R$. This shows that $\mathcal{O}(U) = R$.

but $U \neq \mathrm{Spec}(R)$. $\Rightarrow$ U is not affine.

Some finiteness conditions:

Def X a top space is quasicompact if any open cover has a finite subcover.

Remark Affine schemes are quasi compact: $\{U_\lambda\} \Rightarrow \cap U_\lambda^c = \emptyset$.

$$I(\cap U_\lambda^c) = \sum I(U_\lambda^c) = (1) \Leftrightarrow$$

$1 =$ finite sum of α_λ $\alpha_\lambda \in I(U_\lambda^c)$.

$\Rightarrow$ finite subset of Λ already has $\cap U_\lambda^c = \emptyset$.

Observation A quasi compact schemes are finite unions of affine schemes, and conversely if a scheme is a finite union of affine schemes, then it is quasi-compact.

Def A scheme is noetherian if it is noetherian as a space and is locally spectrum of noetherian ring.

Def A space is noetherian if any decreasing sequence of closed sets stops.
$\Leftrightarrow$ increasing sequence of open sets stops.

Prop noetherian space is quasicompact.

Prop A scheme is noetherian iff it is a finite union of spectra of noetherian rings.

Proof $\Rightarrow$ clear
$\Leftarrow$ clear

Clearly open subscheme of a noetherian scheme is noetherian.

Closed subschemes: Let $Z \subset X$ closed subset define O_Z by: given $U \subset Z$ open
$U \cup Z^c \subset X$ open

Let $V \subset X$ be affine open

$$O_Z(V \cap Z) = O_X(V)/I(Z \cap V).$$

Next ~~the~~ we prove that this gives a well-defined ringed space.

Let's construct projective space:
Several approaches
1) By gluing affine spaces.
2) As "projective spectrum" of a graded ring.

Motivation:
As a set. Let k be a field.

Consider vectors $(x_0, \dots, x_n) \neq (0, \dots, 0) \in k^{n+1}$

up to equivalence relation
$(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$ $\lambda \in k^\times$.

Observation can be covered by sets $U_i = \{(x_0, \dots, x_n) \mid x_i \neq 0\}$.

$U_i \cong k^n$ via

$$(x_0, \dots, x_n) \to \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right)$$

$i = 0, 1, \dots, n$.

T.B.C.