Algebraic geometry

Last time: Construction of projective space by gluing.

U_1, U_2, U_3 $\quad W_{ij} \subset U_i \quad W_{ji} \subset U_j \quad \varphi_{ij} : W_{ij} \to W_{ji}$ isomorphisms

We can glue U_1, U_2 along $W_{12} \simeq W_{21}$. Get $U_1 \cup_{W_{12}} U_2 = V$

To glue V to U_3 we needed:

$$\left(\begin{array}{ccc} W_{12} \cap W_{13} & \xrightarrow{\varphi_{12}} & W_{21} \\ & \searrow \varphi_{13} \quad \nearrow \varphi_{32} & \\ & W_{31} \cap W_{32} & \end{array} \right.$$

we need: 1) $\varphi_{13}(W_{12} \cap W_{13}) = W_{31} \cap W_{32}$

2) $\varphi_{32} \circ \varphi_{13} = \varphi_{12}$

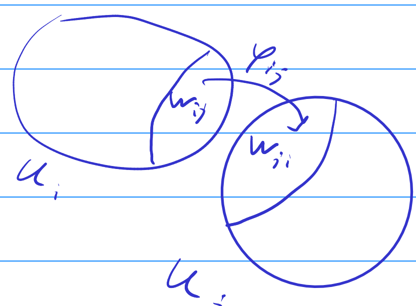
For $\mathbb{P}^n$ $\quad U_i = \operatorname{Spec} k[x_0, \dots, \hat{x}_i, \dots, x_n]$

$U_i \supset W_{ij} = \operatorname{Spec} \underbrace{k[x_0, .., \hat{x}_i, \dots, x_n]\left[\frac{1}{x_j}\right]}_{R_{ij}}$ ← localalize w.r.t. $\{x_j^n \mid n \geq 0\}$

$\varphi_{ij} : W_{ij} \to W_{ji}$

$\varphi_{ij}^* : R_{ji} \to R_{ij}$

$W_{ij} \cap W_{ik} = \operatorname{Spec} k[x_0, , \hat{x}_i, .., x_n]\underbrace{\left[\frac{1}{x_j}, \frac{1}{x_k}\right]}_{\text{means localization}} = R_{ijk}$



$\varphi_{ik}(W_{ij} \cap W_{ik}) = W_{ki} \cap W_{kj}$?

$\varphi_{ik}|_{W_{ij} \cap W_{ik}} = ?$

$\operatorname{Im}\left(\varphi_{ik}|_{W_{ij} \cap W_{ik}}\right) \overset{?}{\underset{(1)}{\subset}} W_{ki} \cap W_{kj} \subset W_{ki}$

$\varphi_{ik} \sim R_{ki} \to R_{ik}$

$W_{ij} \cap W_{ik} \longrightarrow W_{ik} \xrightarrow{\varphi_{ik}} W_{ki}$

$W_{ij} \cap W_{ik} \to W_{ki} \cap W_{kj} \subset W_{ki}$ $(*)$

$(R_{ijk} \xleftarrow{\text{invert } x_j} R_{ik} \xleftarrow{\varphi_{ik}^*} R_{ki}$

$R_{ijk} \xleftarrow{\varphi^*} R_{kij}$, $\quad R_{ki} \xrightarrow{\text{invert } x_j} R_{kij}$

The question: can φ_{ik} be decomposed as in $(*)$?

φ_{ik}^* : R_{ki} (no x_k, x_i inverted) $\to$ R_{ik} (no x_i, x_k inverted

$x_r \longrightarrow \frac{x_r}{x_k}$

$x_i \longrightarrow \frac{1}{x_k}$

$\searrow$ R_{ijk} (no x_i; both x_k, x_j inverted)

$\downarrow$

R_{kij} | no x_k; x_i, x_j inverted $\quad \exists ? \dashrightarrow R_{ijk}$

x_j goes to something invertible?

$x_j \to \frac{x_j}{x_k}$, indeed is invertible. So condition 1) is true.

2) $\varphi_{kj} \circ \varphi_{ik}\big|_{W_{ik} \cap W_{ij}} = \varphi_{ij}$?

$$\begin{array}{ccc} W_{ij} \cap W_{ik} & \xrightarrow{\varphi_{ik}} & W_{ki} \cap W_{kj} \\ & \searrow^{\varphi_{ij}} & \downarrow \varphi_{kj} \\ & & W_{jk} \subset U_j \end{array}$$

The composition $\varphi_{kj} \circ \varphi_{ik}$ is given by

$$\mathcal{O}(U_j) = k[x_0, \dots, \hat{x}_j, \dots, x_n] \xrightarrow{\varphi_{kj}^*} R_{kij} \xrightarrow{\varphi_{ik}^*} R_{ijk}$$

$$\begin{pmatrix} x_r \to \frac{x_r}{x_j} \\ x_k \to \frac{1}{x_j} \end{pmatrix} \qquad \begin{pmatrix} x_r \to \frac{x_r}{x_k} \\ x_i \to \frac{1}{x_k} \end{pmatrix}$$

$r \neq i,j,k$: $\quad x_r \rightsquigarrow \frac{x_r / x_k}{x_j / x_k} = \frac{x_r}{x_j}$

$x_k \rightsquigarrow \frac{x_k}{x_j}$

$x_i \to \frac{x_i}{x_j} \to \frac{1/x_k}{x_j/x_k} = \frac{1}{x_j}$

This is precisely φ_{ij}^*

Proj construction

Our construction of $\mathbb{P}^n$ was a bit ugly.
For $\mathbb{P}^n$ we can consider polynomials in $x_0, \dots, x_n$ (n+1 variables), but we should keep in mind that a polynomial (ex. $x_1 + x_2$) does not define a function on $\mathbb{P}^n$, nevertheless ratio $\frac{P}{Q}$, where P, Q are homogeneous of same degree does define a function, only where $Q \neq 0$. So

Remark 1. It makes sense to say that $Q \neq 0$ at some point (if Q is homogeneous of degree k, then
$Q(\lambda x_0, \dots, \lambda x_n) = \lambda^k Q(x_0, \dots, x_n)$
$\neq 0 \iff \neq 0$ $(\lambda \neq 0)$)

So we want to say that functions of the form $\frac{P}{Q}$ form $\mathcal{O}_{\mathbb{P}^n}(U_{Q\neq 0})$.

Construction of Proj

Input: R a graded ring. This means: R is a ring,
we have a decomposition $R = R_0 \oplus R_1 \oplus \dots = \bigoplus_{k=0}^{\infty} R_k$, and
$R_i \cdot R_j \subset R_{i+j}$ $(i, j = 0, 1, \dots)$
$x \in R$ is called homogeneous if $x \in R_k$, some k, then k is called degree, $\deg x = k$.

Idea: k measures the failure of x to be a true function.

Output: a scheme $\mathrm{Proj}(R)$

1. As a set, $\mathrm{Proj}(R) = \{$homogeneous prime ideals not containing $R_{>0}\}$. Explanation: $I \subset R$ is called homogeneous if $I = \bigoplus_{k \geq 0} I_k$, where $I_k \subset R_k$. (Example: $(x + x^2)$ is not homogeneous
$x + x^2 = \underset{\in R_1}{x} + \underset{\in R_2}{x^2} \in I$, $R = k[x]$ but $x, x^2 \notin I$)

$R_{>0} = \bigoplus_{k \geq 1} R_k$ is a homogeneous ideal.

Example $R = k[x_0, \dots, x_n]$ $R_{>0} = (x_0, \dots, x_n) \leftrightarrow$ point $(0, \dots 0) \in \mathbb{A}^{n+1}$ is a prime ideal, but $\nexists$ no point in $\mathbb{P}^{n+1}$.

2. Topology: base of open sets is $\{U_f \mid f \text{ homogeneous}, \deg f > 0\}$
$U_f = \{p \mid f \notin p\}$. Remark if $R_{>0}$ was a point ε in $\mathrm{Proj}(R)$, then $\varepsilon \notin U_f$ $\forall f$.
$\Rightarrow \{U_f\}$ does not cover $\mathrm{Proj}(R)$.

Check · $U_f \cap U_g = U_{fg}$ ($fg \notin p \iff f \notin p \,\&\, g \notin p$).

· $\forall p \; \exists f$ s.t. $\deg f > 0$, $f \notin p \Rightarrow p \in U_f \Rightarrow \bigcup_f U_f = Proj(R) = X$

Notation $X = Proj(R)$

$R[f^{-1}] := S^{-1}R$, where $S = \{f^k \mid k \geq 0\}$

3. Functions. We need to define $\mathcal{O}_X(U_f)$:

$$\mathcal{O}_X(U_f) = (R[f^{-1}])_0$$

Explanation: $R[f^{-1}]$ is a $\mathbb{Z}$-graded ring,

$R[f^{-1}] = \bigoplus_{k \in \mathbb{Z}} (R[f^{-1}])_k$, $(\ )_i \cdot (\ )_j \subset (\ )_{i+j}$.

Why?

element of $R[f^{-1}]$ looks like $\frac{x}{f^m}$, write x as as sum of homogeneous elements $x = \sum x_k$, $\deg x_k = k$.

$\frac{x}{f^m} = \sum_k \frac{x_k}{f^m}$. Let's say $\frac{a}{f^m}$ is homogeneous of degree $\deg(a) - \deg(f) \cdot m$.

Now $\forall k$ homogeneous elements of $R[f^{-1}]$ form an abelian subgroup $\frac{a}{f^m} + \frac{b}{f^n} \ldots$, $\frac{a}{f^m} \cdot \frac{b}{f^n}$ is homogeneous of degree Σ.

It remains to verify: if

$\frac{a}{f^m} = \frac{b}{f^n} \Rightarrow \deg(a) - m \deg(f) = \deg(b) - n \deg(f)$ (*)

or $\frac{a}{f^m} = 0$. To see that $\frac{a}{f^m} = \frac{b}{f^n} \iff$

$(af^n - bf^m) f^r = 0$, some r.

$\Rightarrow af^{n+r} = bf^{m+r}$, if (*) is not true,

then $\deg af^{n+r} \neq \deg bf^{m+r} \Rightarrow af^{n+r} = bf^{m+r} = 0$.

Equivalently, $\mathcal{O}_X(U_f) = \left\{ \frac{a}{f^m} \,\middle|\, a \text{ is homogeneous}, \deg a = m \cdot \deg f \right\}$.

Example $R = k[x_0, \ldots, x_n]$ $f = x_i$, then

$$\mathcal{O}(U_f) = \left\{ \frac{P(x_0, \ldots, x_n)}{x_i^m} \,\middle|\, P \text{ is homogeneous of degree } m, \ m \geq 0 \right\}$$

Let $\mathcal{O}(U_f) \to R_i = k[x_0, \ldots, \hat{x}_i, \ldots, x_n]$ by setting $x_i \to 1$.

Clearly surjective, in fact we have a section $R_i \to \mathcal{O}(U_f)$ sending Q of degree m to $\frac{Q}{x_i^m}$; so isomorphism.

So we have 1) topology base of, 2) $\mathcal{O}_X(U_f)$ rings.

restriction maps. $U_f \subset U_g$ we need $\mathcal{O}_X(U_g) \to \mathcal{O}_X(U_f)$ a homomorphism

any homogeneous prime $p \not\ni f \Rightarrow p \not\ni g$ $\quad p \ni g \Rightarrow p \ni f$

consider the max. homogeneous ideal containing g, but not f^k all k.

get a contradiction unless $f^k \in (g)$, some k.

$U_f \subset U_g \Rightarrow f^k = gh$, some k, h.

if in some ring f is invertible $\Rightarrow$ g is also invertible, in particular we have a homomorphism $R[g^{-1}] \to R[f^{-1}]$, preserving degrees. ($f^k = gh$ can choose h to be homogeneous).

$R \to R[f^{-1}] \leftarrow g$ is invertible, $\searrow R[g^{-1}]$

in particular we have $R[g^{-1}]_0 \to R[f^{-1}]_0$.

$$\mathcal{O}(U_g) = R[g^{-1}]_0 \qquad \mathcal{O}(U_f) = R[f^{-1}]_0$$

Remains to check the sheaf axiom. (next time)

locally an affine scheme (next time)

break until 12:00.

12:15 PS

One exercise: Given an affine scheme $X = \operatorname{Spec}(R)$ we know that basic open sets U_f are affine schemes. Is converse true? Suppose $U \subset X$ is an affine scheme. Is it of the form U_f?

Question Given $U \subset X$ open. Is it affine?

1) We know $U = U_{f_1} \cup \dots \cup U_{f_k}$, some $f_1, \dots, f_k \in R$

We have U $\quad$ $\mathcal{O}_X(U)$, $\pi : U \to \operatorname{Spec} \mathcal{O}_X(U)$. Is this an isomorphism? U is covered by U_{f_i}, U_{f_i} we understand.

Observation: restricted to U_{f_i} π is an isomorphism.

The only way π is not an isomorphism if π is not surjective.

$\Leftrightarrow$ Open sets of $\operatorname{Spec} \mathcal{O}_X(U)$ corresponding to f_i do not cover $\mathcal{O}_X(U)$. $\Leftrightarrow$ $1 \notin (f_1, \dots, f_k)$

Conclusion to test if U is affine, we need to test if $1 \in (f_1, \dots, f_k)$ in $\mathcal{O}_X(U)$.

Example x, y, u, v $\qquad xy + x^2u + y^2v = 0$ $\qquad U = X \setminus \{x=0, y=0\}$ $= U_x \cup U_y$.

$X = \operatorname{Spec} k[x,y,u,v]/(xy + x^2u + y^2v)$

$X \xrightarrow{x,y} \mathbb{A}^2$

$1 + \frac{xu}{y} + \frac{yv}{x} = 0$

$1 + gx + fy = 0$ $\quad 1 \in (x,y)$.

If $(x,y) \neq (0,0)$ fibers are $\mathbb{A}^1$ $\qquad x(y + xu) = -y^2 v$

if $(x,y) = (0,0)$ fiber is $\mathbb{A}^2$ $\qquad \frac{v}{x} = f = \frac{y + xu}{y^2}$ $\qquad \frac{u}{y} = g = \frac{x + yv}{x^2}$

basic affine $\Rightarrow$ $\{x=0, y=0\}$ is given by 1 equation.