Recall there was a mistake in the definition of morphisms of schemes: If X, Y are affine schemes,
Then we want: $Mor(X,Y) = Hom(\mathcal{O}(Y), \mathcal{O}(X))$, but if Mor = Morphisms of ringed spaces, then we have a surjective map $Mor(X,Y) \to Hom(\mathcal{O}(Y), \mathcal{O}(X))$, but for each $\varphi: \mathcal{O}(Y) \to \mathcal{O}(X)$ besides the induced morphism $i_\varphi: X \to Y$ we have some "extra morphisms" obtained as follows: pick $x \in X$, modify i_φ (as a map) by $i'(x) = y'$, for all other arguments the same as i_φ, where y' is a specialization of $y = i_\varphi(x)$.

Still is continuous (because open nbh's of y' $\subset$ open nbh's of y)

any open set U $\quad y' \in U \Rightarrow y \in U$.

$$\Updownarrow$$

$$y \notin U \Rightarrow y' \notin U$$

$$\Updownarrow$$

$$y \in {}^cU \Rightarrow y' \in {}^cU$$

$$\Updownarrow$$

$\forall$ closed Z $\quad y \in Z \Rightarrow y' \in Z \iff \overline{\{y\}} \ni y'$.

actually this i' is also a morphism of ringed spaces.

Solution 1 Impose an assumption: $\varphi: X \to Y$ is a morphism of schemes if it is a morphism of ringed spaces, and for each pair of affine open $U \subset X$, $V \subset Y$ s.t. $\varphi(U) \subset V$ we have that the morphism $U \to V$ is the morphism induced by $\mathcal{O}(V) \to \mathcal{O}(U)$.

Solution 2 Impose assumption: $\forall x \in X$ the ring homomorphism $\varphi_x: \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ satisfies $\varphi_x^{-1}(m) =$ maximal ideal $\iff \varphi_x(m) \subset m$.

(R,m) (R',m') local rings
$\psi: R \to R'$. Then $\psi^{-1}(m')$ is a prime ideal, $\psi^{-1}(m') \subset m$
$\psi^{-1}(m') = m \Rightarrow \psi(m) \subset m'$
$\psi^{-1}(m') \neq m \Rightarrow \exists t \in m$, $\psi(t) \notin m' \Rightarrow \psi(m) \not\subset m'$

Solution 3
if $x \in X$ goes to $y \in Y$, then $\forall U \ni y$ we have

$$\begin{array}{ccc} \varphi^{-1}(U) & \to & U \\ \cup & & \cup \\ x & \to & y \end{array}$$

the diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}(U) & \to & \mathcal{O}(\varphi^{-1}(U)) \\ \downarrow ev_x & & \downarrow ev_y \\ k_x & \longrightarrow & k_y \end{array}$$

equivalence $2) \Leftrightarrow 3)$

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R' \arrow[d] \\ R/m \arrow[r] & R'/m' \end{tikzcd}$$

commutative $\Rightarrow f \in m \Rightarrow \varphi(f) \in m'$

Functoriality | we have R, R' are rings, then

$$\mathrm{Hom}(R, R') = \mathrm{Mor}(\mathrm{Spec}\, R', \mathrm{Spec}\, R)$$

Suppose φ_1, φ_2 $\quad \varphi_1 : R \to R'$, $\varphi_2 : R' \to R''$

$\varphi_2 \circ \varphi_1 : R \to R''$

$\varphi_1^* : \mathrm{Spec}\, R' \to \mathrm{Spec}\, R \qquad \varphi_2^* : \mathrm{Spec}\, R'' \to \mathrm{Spec}\, R'$

$(\varphi_2 \circ \varphi_1)^* : \mathrm{Spec}\, R'' \to \mathrm{Spec}\, R$

THEN: $\quad (\varphi_2 \circ \varphi_1)^* = \varphi_1^* \circ \varphi_2^*$.

Let's finish Proj!

$R = \bigoplus_{n\geq 0} R_n$ $\quad R_+ = R_{>0} = \bigoplus_{n>0} R_n$, $\mathrm{Proj} = \{ p \subset R \mid p \text{ prime, homogeneous, } p \not\supset R_+ \}$

Let $I \subset R$ be a homogeneous ideal $(I = \bigoplus I_n,\ I_n \subset R_n)$

Let's compute $V(Z(I)) \supset I$

From definition:

$$V(Z(I)) = \bigcap_{\substack{p \text{ prime} \\ \text{homogeneous} \\ p \not\supset R_+ \\ p \supset I}} p$$

we know: $\mathrm{rad}(I) = \bigcap_{p \supset I} p$.

Lemma Let $p \subset R$ be prime. Let $Gr(p) = \sum_{n=0} p \cap R_n$. Then $Gr(p)$ is prime.

$$\mathrm{rad}(I) = \bigcap_{p\supset I} p \supset \bigcap_{p \supset I} Gr(p) = \bigcap_{\substack{p\supset I \\ \text{graded}}} p$$

$p \supset I \Rightarrow Gr(p) \supset I$ $\quad$ on the other hand

$\bigcap_{p\supset I} p \subset \bigcap_{\substack{p\supset I \\ \text{graded}}} p$ is clear. So

Prop I graded $\Rightarrow$

$$\mathrm{rad}(I) = \bigcap_{\substack{p \supset I \\ p \text{ graded}}} p.$$

Note: $\mathrm{rad}(I) \subset V(Z(I)) = \bigcap_{\substack{p\supset I \\ \text{graded} \\ p \not\supset R_+}} p$.

Observation: Suppose $x \in R$ s.t. $xR_+ \subset p \Rightarrow x \in p$ or $R_+ \subset p$

if $p \not\supset R_+ \Rightarrow xR_+ \subset p \Rightarrow x \in p$.

so: $xR_+ \subset V(Z(I)) \Rightarrow x \in V(Z(I))$.

Claim $V(Z(I)) = \{ x \mid xR_+ \subset \mathrm{rad}(I) \}$.

Proof: $xR_+ \subset \mathrm{rad}\, I \Rightarrow xR_+ \subset V(Z(I)) \Rightarrow x \in V(Z(I))$

Conversely, suppose $xR_+ \not\subset \mathrm{rad}(I) \Rightarrow \exists\, y \in R_+$ homogeneous s.t. $xy \notin \mathrm{rad}(I)$

$\exists\, y \in R_+$ homogeneous s.t. $xy \notin \mathrm{rad}(I) \Rightarrow \exists\, p$ prime $p \supset I$

$xy \notin p \Rightarrow x \notin p,\ \underline{y \notin p} \Rightarrow p \not\supset R_+ \Rightarrow x \notin V(Z(I))$ ∎

Def $I \subset R$ graded ideal. Saturation $Sat(I) = \bigcup_{m>0} (I : R_{\geq m}) =$

$= \{ x \in R \mid xR_{\geq m} \subset I \text{, some } m \text{ big enough} \}$.

Claim $V(Z(I)) = \mathrm{rad}(Sat(I))$.

$\mathrm{rad}(Sat(I)) = \{ x \mid x^n R_{\geq m} \subset I \text{, some } m, n \text{ big enough} \}$

$x \in \mathrm{rad}(Sat(I)) \Rightarrow$

$$(x R_{>0})^n \subset I \Rightarrow x \in V(Z(I)). \ldots$$

$$V(Z(I)) = \{x \in R \mid x R_{>0} \subset \mathrm{rad}\, I\}.$$

$x^n R_{>0} \subset \mathrm{rad}\, I$

$\forall y \in R_{>0} \quad x^n y \in \mathrm{rad}\, I$

$(xy)^n \in \mathrm{rad}\, I \Rightarrow xy \in \mathrm{rad}\, I$

$x R_{>0} \subset \mathrm{rad}\, I$, so

$V(Z(I)) = (\mathrm{rad}(I) : R_{>0})$ equals its radical

Suppose $f \in R_{>0}$ homogeneous. $X = \mathrm{Proj}\, R$

$U_f \subset X$ open

Suppose U_f is covered by $\{U_{g_\lambda}\}$ $\{g_\lambda \in R_{>0} \mid \lambda \in \Lambda\}$

algebraically:

1) $U_f \supset U_{g_\lambda}$

2) ${}^c U_f = \bigcap_{\lambda \in \Lambda} {}^c U_{g_\lambda} \iff$

$Z(f) = Z(g_\lambda \mid \lambda \in \Lambda) \Rightarrow f \cancel{R_{>0}} \subset \mathrm{rad}(g_\lambda \mid \lambda \in \Lambda) \iff f \cdot f \in \mathrm{rad}\ldots \quad f \in \ldots$

~~$\forall y \in R_{>0}$~~ $(\cancel{f}y)^n = \sum g_\lambda h_\lambda$ some n.

Consider $O_X(U_f) \to O_X(U_{g_\lambda})$ each λ.

$$O_X(U_f) = R[f^{-1}]_0 \overset{?}{\dashrightarrow} R[g_\lambda^{-1}]_0 = O_X(U_{g_\lambda})$$

1) ${}^c U_f \subset {}^c U_{g_\lambda} \iff V({}^c U_f) \supset V({}^c U_{g_\lambda}) \Rightarrow g_\lambda \cancel{R_{>0}} \subset \mathrm{rad}(f)$.

$\Downarrow$

$g_\lambda^2 \in \mathrm{rad}(f) \Rightarrow g_\lambda \in \mathrm{rad}(f)$.

$g_\lambda \in \mathrm{rad}(f) \Rightarrow$

$g_\lambda^n = f \cdot h \quad h \in R$

$$\frac{a}{f^m} \to \frac{a h^m}{g_\lambda^{mn}}$$

$\frac{a}{f^m} \in R[f^{-1}]_0 \qquad R[g_\lambda^{-1}]_0$ this gives the restriction map.

Lemma $VZ(I) \cap R_{>0} = \mathrm{rad}(I) \cap R_{>0}$

Proof Clearly $\mathrm{rad}(I) \cap R_{>0} \subset VZ(I) \cap R_{>0}$

$\supset$: $x \in VZ(I) \cap R_{>0} \Rightarrow$

$x R_{>0} \in \mathrm{rad}(I) \Rightarrow x^2 \in \mathrm{rad}(I)$

$\Rightarrow x \in \mathrm{rad}(I)$.

2) $f^n = \sum g_\lambda h_\lambda$ this is a finite sum, so only finitely many summands are $\neq 0$.

Remark Hence finitely many U_{g_λ} cover U_f, assume Λ is finite.

Take $\frac{a}{f^m}$; Suppose $\frac{a h_\lambda^m}{g_\lambda^{mn}} = 0$ in $R[g_\lambda^{-1}]_0 \subset R[g_\lambda^{-1}]$. all λ.

we can find $M \gg 0$ s.t. $a h_\lambda^m g_\lambda^M = 0$.

assume $a (h_\lambda g_\lambda)^M = 0$; so $a f^{nM \cdot |\Lambda|} = 0 \Rightarrow$

$\frac{a}{f^m} = 0$

So injectivity is proved.

2nd sheaf condition:

given $\frac{a_\lambda}{g_\lambda^n}$ which agree on $U_{g_\lambda} \cap U_{g_{\lambda'}}$ $(\lambda, \lambda' \in \Lambda)$

we need to construct a function on U_f.

Can assume n is big enough so that

$a_\lambda g_{\lambda'}^n = a_{\lambda'} g_\lambda^n$ $\forall \lambda, \lambda'$. There is an expansion $f^N = \sum h_\lambda g_\lambda^n$ for $N \gg 0$

Consider $\frac{\sum a_\lambda h_\lambda}{f^N}$, on $U_{\lambda'}$: we have $\left(\sum_\lambda a_\lambda h_\lambda\right) g_{\lambda'}^n = \sum_\lambda a_{\lambda'} h_\lambda g_\lambda^n = a_{\lambda'} f^N$, so

$$\frac{\sum a_\lambda h_\lambda}{f^N} = \frac{a_{\lambda'}}{g_{\lambda'}^n}.$$

So the second sheaf condition is true.

$$\boxed{\mathbb{P}^n = \left(\mathbb{A}^{n+1} \setminus \{0\}\right) / G_m}$$

$G_m = \mathbb{A}^1 \setminus \{0\}$ is a group.

Fact

For Affine Scheme $\operatorname{Spec} R$ an action of G_m is the same as a grading on R.

$G_m \times \operatorname{Spec} R \to \operatorname{Spec} R$

$R \to R[t, t^{-1}]$

$R_n \to R_n \cdot t^n$

We proved: functions on $Proj(R)$ form a sheaf.
It remains to check that this is a scheme, more concretely:

U_f is an affine scheme,

$$U_f = \mathrm{Spec}\left(R[f^{-1}]_0\right).$$

From general lemma we have a map of ringed spaces,
$U_f \to \mathrm{Spec}\, R[f^{-1}]_0$. Let's prove that this is an isomorphism.

Points on both sides:
On the left: $A = \mathfrak{p}$ graded prime ideals, $\mathfrak{p} \not\ni f$. (Note that $\mathfrak{p} \not\supset R_+$ is automatic)
On the right: $B =$ prime ideals of $R[f^{-1}]_0$.

$\mathfrak{p} \subset R$ graded prime $\mathfrak{p} \not\ni f \longrightarrow \mathfrak{p}[f^{-1}] \subset R[f^{-1}]$ is also a prime ideal.

$\mathfrak{p}[f^{-1}] \cap R[f^{-1}]_0$ is also a prime ideal.

$$A \longrightarrow B$$

$\pi: R \to R[f^{-1}]$

$B \to A$: $\mathfrak{p} \subset R[f^{-1}]_0$ prime $\to \tilde{\mathfrak{p}} = \mathrm{rad}\, \mathfrak{p} R[f^{-1}]$

$\deg x = a$, $\deg y = b$, $\deg f = k$

$xy \in \tilde{\mathfrak{p}} \Rightarrow (xy)^n \in \mathfrak{p} R[f^{-1}] \Rightarrow (xy)^{nk} \in \mathfrak{p} R[f^{-1}]$

$$\frac{(xy)^{nk}}{f^{n(a+b)}} = \left(\frac{x^{nk}}{f^{an}}\right)\left(\frac{y^{nk}}{f^{bn}}\right) \in \mathfrak{p} R[f^{-1}] \cap R[f^{-1}]_0$$

So $\frac{x^{nk}}{f^{an}}, \frac{y^{nk}}{f^{bn}}$ lie in $R[f^{-1}]_0$, the product is in $\mathfrak{p}$. $\mathfrak{p} R[f^{-1}] = \bigoplus_{m \in \mathbb{Z}} \mathfrak{p} R[f^{-1}]_m$.
hence one of them, say $\frac{x^{nk}}{f^{an}}$ is in $\mathfrak{p}$. $\Rightarrow$ prime
$x^{nk} \in \tilde{\mathfrak{p}} \Rightarrow x \in \tilde{\mathfrak{p}}$. $\tilde{\mathfrak{p}}$ is a graded ideal of $R[f^{-1}]$, corresponds to $\pi^{-1}(\tilde{\mathfrak{p}})$ a graded prime ideal in R.

clearly $\tilde{\mathfrak{p}} \cap R[f^{-1}]_0$ gives back $\mathfrak{p}$.
Conversely, starting with $\mathfrak{p} \subset R \to \left(\mathfrak{p}[f^{-1}]_0\right) \cdot R[f^{-1}]$

$x \in \mathfrak{p}[f^{-1}] \Leftrightarrow x^k \in \mathfrak{p}[f^{-1}] \Leftrightarrow \frac{x^k}{f^{\deg x}} \in \mathfrak{p}[f^{-1}]_0$, this shows that $\widetilde{\mathfrak{p}[f^{-1}]_0} = \mathfrak{p}[f^{-1}]$, from the property of localization $\pi^{-1}\mathfrak{p}[f^{-1}] = \mathfrak{p}$. So the map

$U_f \to \operatorname{Spec} R[f^{-1}]_0$ is a bijection. basic open sets correspond to g s.t. $g^n = fh$, some n

g, g^k define the same open set. $\frac{g^k}{f^{\deg g}} \in R[f^{-1}]_0$, so this is how the bijection between open sets goes.

Note: $k = \deg f$

$$R[g^{-1}]_0 = \left\{ \frac{x}{g^n} \;\middle|\; \begin{array}{l} x \in R \\ \deg x = n \deg g \end{array} \right\}$$

$$R[f^{-1}]_0 = \left\{ \frac{x}{f^n} \right\}$$

the restriction map $\frac{g^k}{f^{\deg g}}$ goes to $\frac{g^k h^{\deg g}}{g^{n \deg g}}$, which is invertible, inverse is $\frac{f^{\deg g}}{g^k}$ in $R[g^{-1}]_0$.

So we have a map

$$(*) \quad R[f^{-1}]_0 \left[\left(\frac{g^k}{f^{\deg g}} \right)^{-1} \right] \to R[g^{-1}]_0.$$

the inverse is provided by sending

$$R[g^{-1}]_0 \ni \frac{x}{g^n} = \frac{x g^{n(k-1)}}{g^{nk}} = \frac{x g^{n(k-1)}}{\left(g^k / f^{\deg g}\right)^n} \cdot \frac{1}{f^{n \deg g}} = \underbrace{\frac{x g^{n(k-1)}}{f^{n \cdot \deg g}}}_{\in R[f^{-1}]_0} \cdot \left(g^k / f^{\deg g} \right)^{-n}$$

So $(*)$ is an isomorphism.

It would be nice if we had a description of $R[f^{-1}]_0$ by a universal property. That would simplify!

Examples. k field

$\text{Proj}(k[x]) \overset{?}{=}$

$\exists$ 2 graded prime ideals

1) $(x) = R_+$ doesn't correspond to a point.

2) (0) corresponds to a point.

how to compute the sheaf of functions?

take any homogeneous element x^m.

$\left(k[x][x^{-m}]\right)_0 := k[x, x^{-1}]_0 = k$.

So $\text{Proj}\, k[x] = \text{Spec}\, k$.

$X = \text{Proj}\left(k[x,y]\right) = ?$

$U_x \cup U_y = X$ because $(x,y) = R_+$, (no prime of Proj containing (x,y)).

$\mathcal{O}(U_x) = k[x,y][x^{-1}]_0 = k[\frac{y}{x}]$ (polynomials in $\frac{y}{x}$)

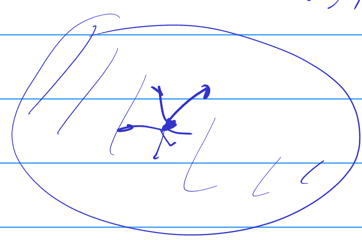
$\mathcal{O}(U_y) =$ polynomials in $\frac{x}{y}$

$\mathcal{O}(U_x \cap U_y) = k[x,y]\left[(xy)^{-1}\right]_0 = k\left[\frac{x}{y}, \frac{y}{x}\right]$ precisely we have $X = \mathbb{A}^1 \cup \mathbb{A}^1$ glued along $\mathbb{A}^1 \setminus \{0\}$ via the map $t \mapsto t^{-1}$.

Analogously $\mathbb{P}^n = \text{Proj}\, k[x_0, \dots, x_n]$. More next time.

For future thoughts. Let $R = k[x,y]$, $m = (x,y) \subset R$

Let $\widetilde{R} = R \oplus m \oplus m^2 \oplus \dots$, $\widetilde{R}_n = m^n$. $\widetilde{R}$ is a graded ring. How to compute $\text{Proj}\, \widetilde{R}$? (this is called the blow up of $(0,0)$ in $\mathbb{A}^2$



$(f_i f_j)^n (a_i f_j - a_j f_i)^n = 0$ By increasing n we can assume $a_i f_j^N - a_j f_i^N$ is <u>nilpotent</u>.

$$1 = \sum f_i^N h_i$$

Let $a = \sum a_i h_i$ then $\underbrace{a = \frac{a_i}{f_i^N} \text{ on } U_{f_i}}$ modulo nilpotents.