Algebraic geometry
Small literature guide

Alg. geometry

Algebraic algebraic geometry everything is algebra: rings, ideals

"part of complex geometry" everything is complex analitye.

mostly these are equivalent.

In these course we follow

Naive approach to alg geometry.

We want to solve systems of polynomial equations

Example
$$\begin{cases} 2x+3y=1 \\ 6x+7y=1 \end{cases} \quad \begin{vmatrix} \times 3 \\ -1 \end{vmatrix}$$

$2y=2$

$\Downarrow$

$y=1$

plug in y

$2x+3=1$

$x=-1$

$$\begin{cases} x^2=4 \\ x^3=8 \end{cases} \quad \begin{vmatrix} x \\ -1 \end{vmatrix}$$

$4x-8=0 \qquad x=2$

Suppose we want to generalize the last one:

$$\begin{cases} P(x)=0 \\ Q(x)=0 \end{cases}$$

P, Q are polynomials

after normalization

$$\begin{cases} x^m + a_{m-1}x^{m-1} + \dots + a_0 = 0 \\ x^n + b_{n-1}x^{n-1} + \dots + b_0 = 0 \end{cases} \quad \begin{vmatrix} -1 \\ x^{m-n} \end{vmatrix}$$

WLOG $m \geq n$

produces new equation of degree $\leq m-1$

We obtain equivalent system

$$\begin{cases} \text{second equation} & \text{degree } n \\ \text{new equation} & \text{degree } \leq m-1 \end{cases}$$

the Σ of degrees is smaller.

Proceed up to new equation is $0=0$.

we are left with 1 equation

$x^k + \dots = 0$ we can take the roots and this is the solution.

Naively Alg. Geo is about manipulations with systems of equations, like above.

More formally (all rings are assumed commutative)

Definitions R is a **ring** if we have:

Operations $+, \cdot : R \times R \to R$, $- : R \to R$,
elements $0, 1 \in R$

satisfying axioms:

1) $+, -, 0$ make R into an abelian group
2) $\cdot, 1$ makes R into a commutative monoid
$1a = a \cdot 1 = a$ $\quad (ab)c = a(bc) \quad ab = ba$
3) $\cdot$ is bilinear w.r.t. the abelian group structure

$(a+b)c = ac + bc$.

Examples: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}[x]$ (polynomials in one variable)

$\mathbb{Z}/p\mathbb{Z}$ (p prime) $\quad \mathbb{Z}/n\mathbb{Z}$ (n integer)

$C(X)$ (X top space, continuous functions)

$\mathbb{Z} \times \mathbb{Z}$ coordinatewise $+, \cdot$ $\quad 1 = (1,1)$, $0 = (0,0)$

Def Some rings are **fields**:
$0 \neq 1$, $\forall x \in R$ $x \neq 0 \Rightarrow \exists y$ s.t. $xy = 1$,

equivalently $R \setminus \{0\}$ is an abelian group for $\cdot$.

Examples $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/p\mathbb{Z}$

Example Let R be a ring
The polynomial ring $R[x]$ is a ring of
elements are expressions $a_0 + a_1 x + \dots + a_n x^n$
(assume $a_n = 0 \Rightarrow a_0 + a_1 \dots + a_n x^n = a_0 + \dots + a_{n-1} x^{n-1}$)
$+ : (a_0 + \dots + a_n x^n) + (b_0 + \dots + b_n x^n) = (a_0 + b_0) + \dots + (a_n + b_n) x^n$
$\cdot : (a_0 + \dots + a_n x^n)(b_0 + \dots + b_m x^m) = a_0 b_0 + (a_1 b_0 + a_0 b_1) x + (a_2 b_0 + a_1 b_1 + a_0 b_2) x^2 + \dots$

Equivalently:
$R[x]$ = set of infinite sequences $(a_0, a_1, \dots \in R)$ such that $a_i = 0$ ($i > N$ some N)

$+$: coordinatewise

$\cdot$: $(a_0, a_1, \dots) \cdot (b_0, b_1, \dots) = (c_0, c_1, \dots)$

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

$0 = (0, 0, 0, \dots)$
$1 = (1, 0, 0, \dots)$

Remark Polynomials are not functions,
for example R finite $R \neq 0$ then
$\underset{1}{\underset{\parallel}{(1, 0, \dots)}}$ $\quad \underset{x}{\underset{\parallel}{(0, 1, 0, \dots)}}$ $\quad \underset{x^2}{\underset{\parallel}{(0, 0, 1, \dots)}}$
are all different, but $\exists$ only finitely many functions $R \to R$.

Def polynomials in finitely many variables $x_1, \dots, x_n$:

$$\underbrace{R[x_1]}_{\text{a ring}}[x_2] \dots [x_n] = R[x_1, \dots, x_n].$$

A System of polynomial equations (over R) in n variables is a collection $(f_1, \dots, f_m) \in R[x_1, \dots, x_n]$

Which systems are equivalent?
There is a naive definition "procedural".

More abstract definition:
"A system of equations" = ideal.

Def Ideal $\mathfrak{a} \subset R$ is a subset satisfying:
1) $x, y \in \mathfrak{a} \Rightarrow x+y, x-y \in \mathfrak{a}$, $0 \in \mathfrak{a}$
2) $x \in \mathfrak{a}$ $y \in R \Rightarrow xy \in \mathfrak{a}$. $(\mathfrak{a} R \subset \mathfrak{a})$

Ideals vs. systems of equations:

$\Leftarrow$ $f_1, \dots, f_m$ corresponds to the ideal generated by $f_1, \dots, f_m$:

$$\mathfrak{a} = \{ f \in R \mid \exists g_1, \dots, g_m \in R \text{ so that } f = f_1 g_1 + f_2 g_2 + \dots + f_m g_m \}$$

$\mathfrak{a}$ = "all equations which follow from $f_1, \dots, f_m$".

$\Rightarrow$ $\mathfrak{a} \Rightarrow$ infinite system of equations given by all elements of $\mathfrak{a}$.

Future theorem: for a nice ring R any ideal can be generated by a finite sequence $f_1, \dots, f_m$.

Def 2 systems $f_1, \dots, f_m \in R[x_1, \dots, x_n]$
$f_1', \dots, f_{m'}' \in R[x_1, \dots, x_n]$
are equivalent if

$$\underset{\substack{\text{the ideal} \\ \text{generated by} \\ f_i}}{(f_1, \dots, f_m)} = \underset{\substack{\text{the ideal} \\ \text{generated by } f_i'}}{(f_1', \dots, f_{m'}')} \subset R[x_1, \dots, x_n]$$

Example $(1) = R$

Prop Suppose $(f_1, \dots, f_m) = (1)$, $R \neq \{0\}$, then the system has no solutions:

Pf $1 = \sum_{i=1}^{m} f_i g_i \Rightarrow$ if $x_1^{(0)}, \dots, x_n^{(0)} \in R$ is a solution
$f_i(x_1^{(0)}, \dots, x_n^{(0)}) = 0$ $(i = 1, \dots, m)$
we obtain $1 = 0$
contradiction.

Remark the opposite implication is not true in general $x^2 + 1 = 0$ over $\mathbb{R}$
no solutions, $1 \notin (x^2+1) \cdot$ polynomial

Ideals vs sets of solutions:

Notations: $\mathfrak{a} \subset R[x_1, \dots, x_n]$ define $Z(\mathfrak{a})$

$$Z(\mathfrak{a}) = \{ (x_1^{(0)}, \dots, x_n^{(0)}) \in R^n \text{ s.t. } \forall f \in \mathfrak{a} \text{ we have } f(x_1^{(0)}, \dots, x_n^{(0)}) = 0 \} \subset R^n$$

called the zero set

conversely, if $S \subset R^n$ is an arbitrary set, then

$$V(S) = \{ f \in R[x_1, \dots, x_n] \mid f(x_1^{(0)}, \dots, x_n^{(0)}) = 0 \ \forall (x_1^{(0)}, \dots, x_n^{(0)}) \in S \}$$

Prop $V(S) \subset R$ is an ideal.

Prop $S_1 \subset S_2 \Rightarrow V(S_2) \subset V(S_1)$

$\mathfrak{a}_1 \subset \mathfrak{a}_2 \Rightarrow Z(\mathfrak{a}_2) \subset Z(\mathfrak{a}_1)$

$V(Z(\mathfrak{a})) \supset \mathfrak{a}$ (not $=$ in general, for instance if $Z(\mathfrak{a}) = \emptyset$ $V(Z(\mathfrak{a})) = V(\emptyset) = (1)$)

$Z(V(S)) \supset S$ (not $=$ in general)
Example
$S = \{1, 2, 3, \dots\} \subset \mathbb{R}$
$V(S) = (0)$ $\quad Z(V(S)) = Z((0)) = \mathbb{R}$

Construction of a quotient ring

Input: R ring, $\mathfrak{a} \subset R$ ideal

Output: new ring $R/\mathfrak{a}$

as a set, as an abelian group $R/\mathfrak{a}$ = quotient of abelian groups

elements of $R/\mathfrak{a}$ are written as $f+\mathfrak{a}$ for $f \in R$

or $[f]$ or $\overline{f}$. We say $f+\mathfrak{a} = g+\mathfrak{a}$ if

$f-g \in \mathfrak{a}$ (also write $f \equiv g \pmod{\mathfrak{a}}$)

$(f+\mathfrak{a})(g+\mathfrak{a}) = (fg+\mathfrak{a})$

Prop This is well-defined:

if $f+\mathfrak{a} = f'+\mathfrak{a}$, then

$fg - f'g = (\underbrace{f-f'}_{\in \mathfrak{a}})g \in \mathfrak{a} \Rightarrow fg+\mathfrak{a} = f'g+\mathfrak{a}$.

Prop with this product $R/\mathfrak{a}$ is a ring

$(f_1+f_2+\mathfrak{a})(g+\mathfrak{a}) = f_1g+f_2g+\mathfrak{a} = (f_1g+\mathfrak{a})+(f_2g+\mathfrak{a})$

$\Downarrow$ product is bilinear.

Examples $(m) \subset \mathbb{Z}$ ideal generated by $m \in \mathbb{Z}$

form $\mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/(m)$ we have seen before.

Universal property

Observation: R, R' rings

if $\varphi : R \to R'$ is a homomorphism

$\varphi(xy) = \varphi(x)\varphi(y)$

$\varphi(x+y) = \varphi(x)+\varphi(y)$

$\varphi(0) = 0 \quad \varphi(1) = 1$

then $\varphi^{-1}(0) = \ker\varphi$ is an ideal.

$f \in \ker\varphi \quad g \in R \Rightarrow$

$\varphi(fg) = \varphi(f)\varphi(g) = 0\cdot\varphi(g) = 0$

$\Rightarrow fg \in \ker\varphi$

Obs 2: $\exists$ projection map $R \to R/\mathfrak{a}$

it is a homomorphism ($fg+\mathfrak{a} = (f+\mathfrak{a})(g+\mathfrak{a})$)

Prop $R \to R/\mathfrak{a}$ is universal among homomorphisms $\varphi\ R \to R'$ satisfying $\varphi(\mathfrak{a}) = 0$ ($\mathfrak{a} \subset \ker\varphi$).

Equivalently:

1) $\pi : R \to R/\mathfrak{a}$ satisfies $\mathfrak{a} \subset \ker(\pi)$ (because $\ker(\pi) = \mathfrak{a}$)

2) if $\varphi : R \to R'$ also satisfies $\mathfrak{a} \subset \ker\varphi$

then $\exists!\ \psi : R/\mathfrak{a} \to R'$ which makes the diagram commutative:

$$\begin{tikzcd} R \arrow[rr, "\forall\varphi"] \arrow[dr, "\pi"'] & & R' \\ & R/\mathfrak{a} \arrow[ur, dotted, "\exists!\,\psi"'] & \end{tikzcd}$$

$\varphi = \psi \circ \pi$.

Prove Uniqueness: π surjective, so if

$\psi(\pi(f)) = \varphi(f) = \psi'(\pi(f))$

$\psi = \psi'$ on $\operatorname{Im}\pi = R/\mathfrak{a} \Rightarrow$

$\psi = \psi'$ everywhere.

existence: Define ψ by $\psi(f+\mathfrak{a}) = \varphi(f)$.

this is well-defined by: $f+\mathfrak{a} = f'+\mathfrak{a} \Rightarrow$

$f-f' \in \mathfrak{a} \quad \varphi(f') = \varphi(f)+\varphi(\underbrace{f'-f}_{\in\mathfrak{a}}) = \varphi(f)$.

ψ is a homomorphism:

$\psi(f+\mathfrak{a}+g+\mathfrak{a}) = \psi(f+g+\mathfrak{a}) = \varphi(f+g) = \varphi(f)+\varphi(g)$

$= \psi(f+\mathfrak{a})+\psi(g+\mathfrak{a})$

$\psi((f+\mathfrak{a})(g+\mathfrak{a})) = \psi(fg+\mathfrak{a}) = \varphi(fg) = \varphi(f)\varphi(g)$

$= \psi(f+\mathfrak{a})\psi(g+\mathfrak{a})$.

Application Suppose $R \supset \mathfrak{a}$ ideal

Suppose $\varphi : R \to R'$ is such that

$\ker\varphi = \mathfrak{a}$, φ is surjective. Then

R' is isomorphic to $R/\mathfrak{a}$:

by the universal property

$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & R' \\ & R/\mathfrak{a} \arrow[ur, "\psi"'] & \end{tikzcd}$$

ψ is surjective because φ is.

ψ is injective because

$\psi(\pi(f)) = 0 \Rightarrow \varphi(f) = 0 \Rightarrow f \in \mathfrak{a} \Rightarrow \pi(f) = 0$.

Functor of points in alg. geometry.

A category C is a collection of Objects $Ob(C)$,

$\forall$ objects X, Y in $Ob(C)$ we have a collection of Morphisms $Mor(X,Y)$ + :

1) $\forall X$ there is a Identity $Id_X \in Mor(X,X)$
2) For X, Y, Z in $Ob(C)$ we have a composition law $\underset{f}{Mor(X,Y)} \times \underset{g}{Mor(Y,Z)} \to \underset{g \circ f}{Mor(X,Z)}$

Satisfying axioms:

1) For X, Y, Z, W $f \in Mor(X,Y)$ $\left(\text{notation: } f : X \to Y\right)$
$f: X \to Y$, $g: Y \to Z$, $h: Z \to W$, then
$h \circ (g \circ f) = (h \circ g) \circ f$
2) $f: X \to Y$ then $f \circ Id_X = Id_Y \circ f = f$.

Def X is isomorphic to Y $X \cong Y$ if
$\exists f: X \to Y$ $g: Y \to X$ s.t. $f \circ g = Id_Y$
$g \circ f = Id_X$.

Examples

Sets = Objects are sets
Morphisms are maps
composition is usual.

Top = topological spaces
continuous maps

Ring = rings (commutative, with 1)
homomorphisms

Idea For a fixed ring R consider all elements of $Mor(R, R')$ (all rings R').

This is an example of a functor.

Def C, C' categories, a functor $F: C \to C'$ is a: 1) map $Ob(C) \to Ob(C')$
$X \to F(X)$

2) $\forall X, Y$ a map $Mor(X,Y) \to Mor(F(X), F(Y))$

Satisfying:

1) $f: X \to Y$, $g: Y \to Z$: $F(g \circ f) = F(g) \circ F(f)$
2) X : $F(Id_X) = Id_{F(X)}$.

Fix R Construct a functor F_R : Rings $\to$ Sets

For any ring R' $F_R(R') = Mor(R, R')$

Suppose $f: R' \to R''$ $F_R(f) : Mor(R, R') \to Mor(R, R'')$
sends $g: R \to R'$ to $f \circ g : R \to R''$

Axioms:

$F_R(Id_{R'})$: $Mor(R, R') \to Mor(R, R')$
$g: R \to R'$ to g

$F_R(g \circ f)$: $Mor(R, R') \to Mor(R, R''')$
$f: R' \to R''$ $h: R \to R' \to (g \circ f) \circ h$
$g: R'' \to R'''$ $\qquad \qquad \qquad \qquad \| \text{ obviously}$
$\left(F_R(g) \circ F_R(f)\right)(h) = g \circ (f \circ h)$

So $F_R(g \circ f) = F_R(g) \circ F_R(f)$.

The way to think about F_R is.

Think

$$R = \mathbb{Q}[x_1,..,x_n]/(f_1,..,f_m)$$

Suppose R' is another ring.

$F_R(R') = ?$ set of homomorphisms

$\varphi: R \to R'$

$$\begin{tikzcd} \mathbb{Q}[x_1,..,x_n] \arrow[r] \arrow[d, "\pi"] & R' \\ R \arrow[ur] & \end{tikzcd}$$

by the universal property of quotient rings

$$F_R(R') = \{ \underbrace{\varphi: \mathbb{Q}[x_1,..,x_n] \to R'}_{\text{Next understand this set.}} \mid \varphi(f_i) = 0 \ (i = 1,..,m) \}$$

Prop $\quad \mathrm{Mor}(\mathbb{Q}[x_1,..,x_n], R') = \{ (\varphi_0, g_1,..,g_n) \mid \varphi_0 : \mathbb{Q} \to R', \ g_i \in R' \}$

Proof $\quad$ Given $\varphi: \mathbb{Q}[x_1,..,x_n] \to R'$

define φ_0 by restricting φ to $\mathbb{Q} \subset \mathbb{Q}[x_1,..,x_n]$

g_i by $g_i = \varphi(x_i)$.

Conversely, given $\varphi_0, g_1,..,g_n$

Define φ by $\varphi(\sum \underset{\in \mathbb{Q}}{c_{i_1,..,i_n}} x_1^{i_1} \cdots x_n^{i_n}) = \sum \varphi_0(c_{i_1 .. i_n}) g_1^{i_1} \cdots g_n^{i_n}$ $\square$

Cor $\quad F_R(R') =$ set of pairs (φ_0, g) $\{ \varphi_0 : \mathbb{Q} \to R'$, and g a solution to the system $f_1,..,f_m$ in R' $\}$.

We will show that the F_R (called the functor of points) completely determines R).

Example $\quad \mathbb{R}$, equation $x^2 + 1 = 0$

corresponds to ideal $(x^2+1) \subset \mathbb{R}[x]$

quotient ring: $\mathbb{R}[x]/(x^2+1)$

we don't have solutions over $\mathbb{R}$, but we have over $\mathbb{C}$!

$$\mathbb{R}[x]/(x^2+1) \to \mathbb{C}$$

$\varphi_0 : \mathbb{R} \to \mathbb{C}$ is the standard map

$x \mapsto i$.

By the way: $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$.

Main property of F_R

Yoneda Lemma Consider the map

Rings to Fun(Rings, Sets) given by

$R \to F_R$
(contravariant)

Claim 1 This is a functor from Rings to the category with objects functors (Rings, Sets), morphisms are natural transformations:

For 2 functors F: Rings $\to$ Sets
F': —||—

a natural transformation is a map
$\forall X \in$ Rings $F(X) \to F'(X)$ satisfying:

$\forall X, X'$ $\varphi: X \to X'$

$$\begin{array}{ccc} F(X) & \to & F'(X) \\ {\scriptstyle F(\varphi)}\downarrow & & \downarrow {\scriptstyle F'(\varphi)} \\ F(X') & \to & F'(X') \end{array}$$

is commutative

(think $F \to$ system of equations (1)
F' another system of equations (2))

composition of nat. trans-s.
$F(X) \to F'(X) \to F''(X)$ we can compose.

The Lemma $Mor(R, R') \longrightarrow Nat.transf(F_{R'}, F_R)$

is a bijection.

Corollary if F_R is isomorphic to $F_{R'}$,

then R is isomorphic to R'.

Proof F_R isomorphic to $F_{R'}$ $\Rightarrow$ $f: F_R \to F_{R'}$ some natural transformation $g: F_{R'} \to F_R$, $f \circ g = Id_{F_{R'}}$
$g \circ f = Id_{F_R}$

$\Rightarrow$ by the lemma $\exists$ $\tilde{f}: R' \to R$
$\tilde{g}: R \to R'$
$\tilde{f} \circ \tilde{g}$ - goes to $Id_{F_{R'}}$ $\Rightarrow$
$\tilde{f} \circ \tilde{g} = Id_{R'}$ similarly for $\tilde{g} \circ \tilde{f} = Id_R$