Which examples did we have?

$Proj(k[x]) = \text{point}$

$Proj(k[x,y]) = \mathbb{P}^1$ — $\deg x, y = 1$

More generally $Proj(k[x_0,\dots,x_n]) = \mathbb{P}^n$ — $\deg x_i = 1$

Example $X = Proj\ k[x,y]$ $\deg x = 2$ $\deg y = 3$

$(x,y) = k[x,y]_{>0}$. So $X = U_x \cup U_y$.

(0) (x) (y) $(x^3 + \alpha y^2)$

$U_x = Spec\ \underbrace{k[x,y,x^{-1}]_0}$

has basis $x^a y^b$ s.t. $2a+3b=0$, $b \geq 0$

$\Rightarrow b$ is even $a = -\frac{3}{2}b$

So this ring is the polynomial ring in $x^{-3}y^2$, $U_x = \mathbb{A}^1$

Similarly $U_y = \mathbb{A}^1$,

$U_y = Spec\ k\underbrace{[x,y,y^{-1}]_0}_{\text{polynomial ring in } x^3y^{-2}}$

$x^3 y^{-2} = t$

$U_{xy} = \underbrace{\text{Laurent polynomial ring}}_{t^{-1}k[t]}$ in t.

So U_x and U_y are glued along U_{xy} $t \to t^{-1}$

So $X = \mathbb{P}^1$.

Remark R graded ring, if R is Noetherian, then $R_{>0}$ is generated by finitely many elements $f_1, \dots, f_m$. Then U_{f_i} (basic open set) cover $Proj(R)$.

So $Proj(R)$ can be obtained by gluing these affine schemes U_{f_i}.

Next example

The same for $\deg x = m$ $\deg y = n$:

1) if $(m,n) \neq 1$ Let $g = (m,n)$. all elements of R have degree divisible by g. We can divide degree by g: $R' = \bigoplus_{k \geq 0} R_{kg}$.

$\uparrow$ degree $= k$

This gives the same space. WLOG: $g = 1$.

2) $g = 1$ we have $\mathbb{P}^1$ again.

$k[x, y, z]$ $\deg x = a$ $\deg y = b$ $\deg z = c$

WLOG $\gcd(a, b, c) = 1$. Say $\underset{a}{2}, \underset{b}{3}, \underset{c}{5}$.

These are called

weighted projective spaces.

Blow up $R_0 = k[x, y]$ $m = (x, y)$

$R = \underset{\deg = 0}{R_0} \oplus \underset{\deg = 1}{m} \oplus m^2 \dots$ Rees construction

(for proof that in a Noetherian local ring $\bigcap_k m^k = \{0\}$)

$m^k = (x^k, x^{k-1}y, \dots, y^k)$

$R_{>0}$ is generated by $\bar{x}, \bar{y} \in m$ (to distinguish from $x, y \in R_0$)

(b.t.w $R = k[x, y, \bar{x}, \bar{y}] / \bar{x} \cdot y - x \cdot \bar{y}$

$\underset{R_0}{k[x, y]} \oplus (\underset{R_1 (= m)}{\cdot}$

basis of m: $x^a y^b$ $a > 0$ or $b > 0$, $m = x \cdot R_0 \oplus y k[y]$

Clearly $\bar{x} \cdot y = x \cdot \bar{y}$ in R. So there is a surjective homomorphism $k[x, y, \bar{x}, \bar{y}] / \bar{x} \cdot y - x \cdot \bar{y} \to R$.

basis of R_m is given by: $\boxed{m^k = x^k \cdot R_0 \oplus x^{k-1} y \cdot k[y] \oplus \dots y^k k[y]}$

$R_k = \bar{x}^k \cdot R_0 \oplus \bar{x}^{k-1} \bar{y} \cdot k[y] \oplus \dots \bar{y}^k \cdot k[y]$.

Therefore $R = k[x, y, \bar{x}] \oplus k[y, \bar{x}, \bar{y}] \cdot \bar{y}$. $(*)$

Let $R \to k[x, y, \bar{x}, \bar{y}] / (\bar{x} y - x \bar{y}) = R'$ be the map

defined by writing any $f \in R$ as $f_1 + \bar{y} f_2$

$f_1 \in k[x, y, \bar{x}]$, $f_2 \in k[y, \bar{x}, \bar{y}]$, sending it to $f_1 + \bar{y} f_2$ in R'.

$R \to R' \to R$ is the identity $R' \to R \to R'$. We need to show that any element of R' can be written as in $(*)$.

in R' if a monomial is divisible by both $x, \bar{y}$, we can replace $x\bar{y}$ by $\bar{x} y$.

Anton Mellit updated his status.
January 9, 2017 · Fasa, Iran ·

Suppose A is an abstract algebra (or module, or some other structure), by which I mean an algebra given by generators and relations. Suppose B is a concrete algebra, by which I mean an algebra given by a certain explicit vector space with operations. Suppose you want to prove that A is isomorphic to B. This is what you should do:

1) Construct a homomorphism phi from A to B. This means you should say where each generator of A goes and check that each relation of A holds in B.

2) Prove that phi is surjective. This means you should show how arbitrary element of B can be expressed in terms of your generators, using the structure of B.

3) Prove that phi is injective. This means you should construct a candidate basis of A, show that every element of A can be expressed as a linear combination of the basis elements using the structure of A, and show that the images of the basis elements are linearly independent in B.

The step 3 is usually the most difficult. Note that steps 1,2 involve computations in B, while step 3 involves computations in A, provided that it is somehow clear that the images of the basis elements in B are linearly independent. Also note that these three steps are independent in the sense that you can hardly use results from one step to make another step easier.

It is surprising how often this is messed up in math papers. Basically the authors feel that something needs to be proved, but what they write is irrelevant. For instance, step 3 simply repeats the arguments of step 2 in a hope that the result somehow follows. And referees don't seem to notice that some nonsense is written.

So we have 1) $R = k[x,y,\bar{x},\bar{y}]/(\bar{x}y - x\bar{y})$.
2) (x)

$\mathrm{Proj}(R)$ is covered by affine schemes $U_{\bar{x}}$ and $U_{\bar{y}}$. $\begin{pmatrix} \deg \bar{x} = \deg \bar{y} = 1 \\ \deg x = \deg y = 0 \end{pmatrix}$

$$\mathcal{O}_{U_{\bar{x}}} = \left[k[x,y,\bar{x},\bar{y},\bar{x}^{-1}]/(\bar{x}y - x\bar{y}) \right]_0$$

replace $\bar{x}y - x\bar{y}$ by $y = \bar{x}^{-1}x\bar{y}$, so

$$\mathcal{O}_{U_{\bar{x}}} = \underline{k[x,\bar{x},\bar{y},\bar{x}^{-1}]_0} = k[x, \underset{u}{\overset{}{\bar{y}/\bar{x}}}] = k[x,u]$$

So $U_{\bar{x}} = \mathbb{A}^2$.

Similarly $\mathcal{O}_{U_{\bar{y}}} = k[y,v]$ $\quad (v = \bar{x}/\bar{y})$. $\quad U_{\bar{y}} = \mathbb{A}^2$.

Gluing: $\mathcal{O}_{U_{\bar{x}\bar{y}}} = k[x,u,u^{-1}] \cong k[y,v,v^{-1}]$.

$y = \bar{x}^{-1}x\bar{y} = xu$

$v = u^{-1}$

So the gluing homomorphism $k[y,v,v^{-1}] \to k[x,u,u^{-1}]$

is given by $y \to xu \quad v \to u^{-1}$

inverse $\quad yv \leftarrow x$

$\quad v^{-1} \qquad u$

Check: $y \to xu \to yv \cdot v^{-1} = y$.

Geometric picture of the Blow up:

$$X = U_{\bar{x}} \cup U_{\bar{y}} \xrightarrow{(x,y)} \mathbb{A}^2$$

Look at $U_{\bar{x}} \to \mathbb{A}^2 = \mathrm{Spec}\, k[x,y]$

corresponds to $x \to x$, $y \to xu$

On points

(x,u) goes to (x, xu)

$U_{\bar{x}}$ $\quad \cdot (x_0, y_0/x_0)$

$\downarrow$

$\cdot (x_0, y_0)$

$\mathbb{A}^2 \quad (0,0)$

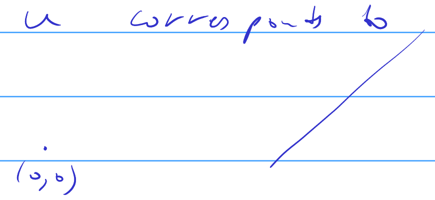
(More generally any element of R_0 defines a morphism $\mathrm{Proj}(R) \to \mathbb{A}^1$)

if $x \neq 0$ then $\exists$ unique point over it in $U_{\bar{x}}$. $(x_0, y_0/x_0)$.

if $x_0 = 0$ $\quad y_0 = 0 \Rightarrow u$ is arbitrary so $\mathbb{A}^1 \overset{x=0}{\subseteq} U_{\bar{x}}$ is mapped to $(0,0)$

$x = x_0 \qquad xu = y_0$

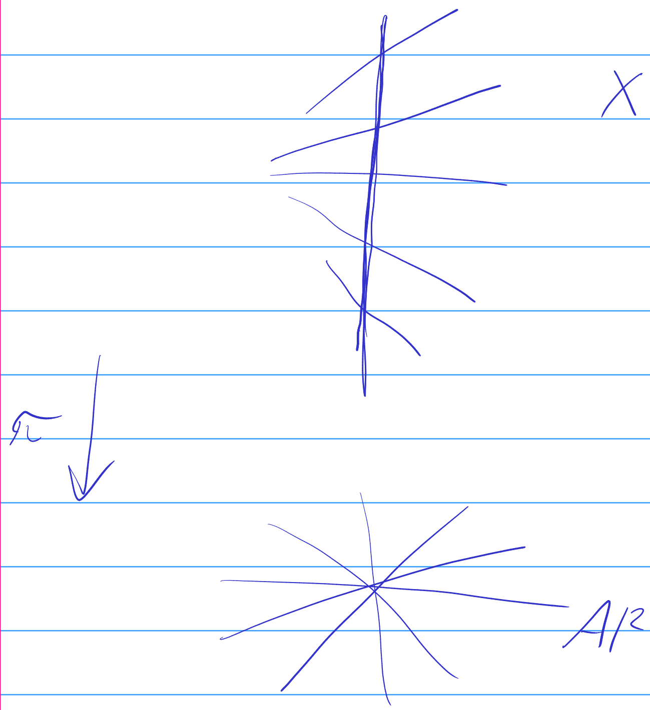
So we can think that u corresponds to the "angle" at which a point goes to 0 on $\mathbb{A}^2$



X

More completely:
preimage of any point $\neq (0,0)$ is exactly 1 point,
the preimage of $(0,0)$ is $\mathbb{P}^1$

$\mathbb{A}^2$



More precisely,

$$\pi^{-1}\left(\mathbb{A}^2 \setminus \{(0,0)\}\right)$$
$$\downarrow \pi$$
$$\mathbb{A}^2 \setminus \{(0,0)\}$$

is an isomorphism.

<u>Pf</u> Check on $U_{\bar{x}}, U_{\bar{y}}$.

<u>Notation</u> $Bl_{(0,0)}\mathbb{A}^2$

blow up this point.

↖ <u>here</u>

Very general: R ring I ideal

$\mathrm{Proj}(R \oplus I \oplus I^2 \ldots)$ usually complicated

$= Bl_{Z(I)} \operatorname{Spec} R$.

<u>Uses</u> (Hironaka) Theorem by blowing sufficiently many times you can get a manifold. "Resolution of singularities".

<u>Physically</u> we want to encode the angle.

More examples (classical algebraic geometry).

Consider $\mathbb{P}^2$ (over some k).

$\mathbb{P}^2 = \operatorname{Proj} k[x,y,z]$ any homogeneous polynomial $f \neq 0$ defines an ideal (f). We want to understand $Z((f)) \subset \mathbb{P}^2$.

Clearly things depend on $\deg f$.

1) $\deg f = 1$ $f = \alpha x + \beta y + \gamma z$ $(\alpha,\beta,\gamma) \neq (0,0,0)$

this is just equation of $\mathbb{P}^1 \subset \mathbb{P}^2$.

Proof WLOG $\alpha \neq 0$, $\alpha = 1$, change equation to

$$x = -\beta y - \gamma z. \qquad k[x,y,z]/(f) \cong k[y,z].$$

Remark. changing x, y, z by a linear substitution doesn't change geometry.

So case 1) is equivalent to $f = x$.

2) $\deg f = 2$ $f = Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2$

(6 parameters, in fact 5 up to scaling).

It is convenient if $2 \neq 0$ in k form a matrix

$$M = \begin{pmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{pmatrix} \quad \text{then} \quad (x\,y\,z)\, M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = f$$

changing $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to $G \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ for some 3×3 invertible matrix G changes M to $G^t M G$.

So $r = \operatorname{rank}(M)$ is invariant.

Suppose $r = 1$: we can choose G so that the second and third columns of MG are 0.

Then $G^t M G$ also has this property. Moreover $G^t M G$ is symmetric, so replace M by $G^t M G$ and

$$M = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ rescaling assume } \alpha = 1.$$

$f = x^2$ $\qquad R/(x^2) = k[x,y,z]/(x^2)$.

What is $\mathrm{Proj}\left(k[x,y,z]/(x^2)\right)$? $\quad U_x, U_y, U_z$

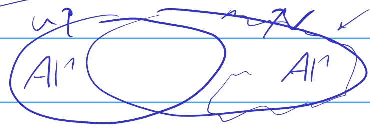
$U_x = \mathrm{Spec}\ k[x,y,z]/x^2\,[x^{-1}] = \mathrm{Spec}\{0\} = \emptyset$

$x/y = u \quad z/y = \tilde{z}$

$U_y \quad : \quad \mathrm{Spec}\left(k[x,y,y^{-1},z]/x^2\right)_0 = k[u,\tilde{z}]/u^2$

$U_z \quad \dots \quad k[v,\tilde{y}]/v^2 \quad v = \frac{x}{z}$

gluing $\tilde{y} = \tilde{z}^{-1}$ $\quad u = v\tilde{z}$. So this is a "thickening" of $\mathbb{P}^1$ w/ nilpotent variables whose ^{2} is



0 glued on the intersection in some interesting way.

Note: Alg. geometry gives a little more information than naively solving $x^2=0$.

Next, $r=2$

Similarly we can transform M to

$$\begin{pmatrix} A & B/2 & 0 \\ B/2 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad f = Ax^2 + Bxy + Cy^2 = 0$$

if $k = \bar{k}$

$A \neq 0$:

$$\left(Ax^2 + Bxy + \frac{B^2}{4A}y^2\right) + \left(C - \frac{B^2}{4A}\right)y^2 = 0$$

$$\left(\sqrt{A}\,x + \frac{B}{2\sqrt{A}}y\right)^2 + \left(\sqrt{C - \frac{B^2}{4A}}\,y\right)^2 = 0$$

note $\det\begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} = AC - \frac{B^2}{4} \neq 0 \implies \neq 0$

$$f = \left(\sqrt{A}\,x + \frac{B}{2\sqrt{A}}y - \sqrt{-1}\sqrt{C - \frac{B^2}{4A}}\,y\right)\left(\sqrt{A}\,x + \frac{B}{2\sqrt{A}}y + \sqrt{-1}\dots\right)$$

by change of basis, $f = x \cdot y$

if $A = 0$ even easier.

So WLOG $f = xy$

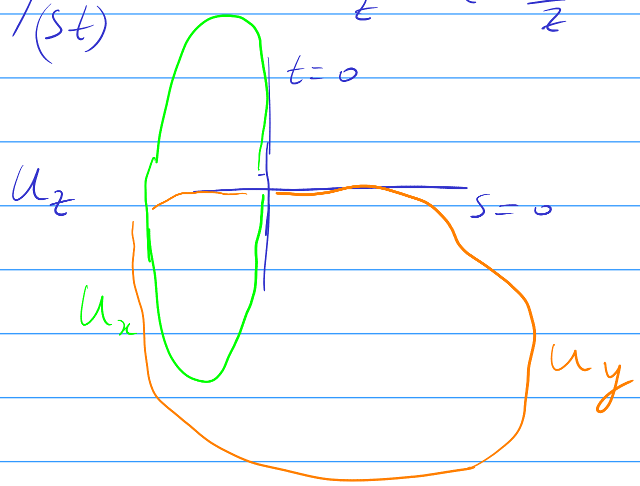
$R = k[x,y,z]/(xy)$

$U_x = \mathrm{Spec}\, k[x, x^{-1}, z]_0$
$= \mathrm{Spec}\, k[u]$ $u = \frac{z}{x}$

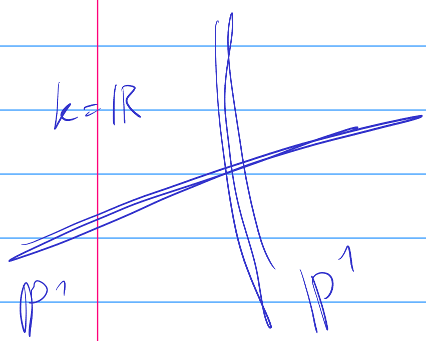
$U_y = \mathrm{Spec}\, k[v]$ $v = \frac{z}{y}$

$U_z = \mathrm{Spec}\left(k[x,y,z,z^{-1}]/(xy)\right)_0$

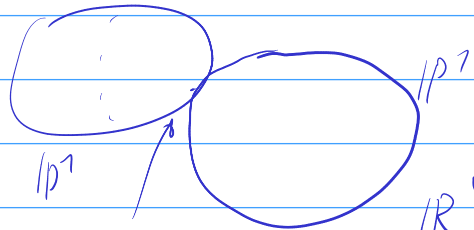
$= \mathrm{Spec}\, k[s,t]/(st)$ $s = \frac{x}{z}$ $t = \frac{y}{z}$



$k = \mathbb{R}$



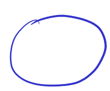
$k = \mathbb{C}$



Finally $r=3$ over $k = \bar{k}$. Any quadratic form can be made diagonal. Another way: choose a vector v s.t. $v^t M v \neq 0$. Let $U = \ker(v^t M)$ choose a basis of U, e_1, e_2, we obtain v, e_1, e_2 basis in which M looks like

$\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$ can be transformed to $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$x^2 + y^2 + z^2 = 0$

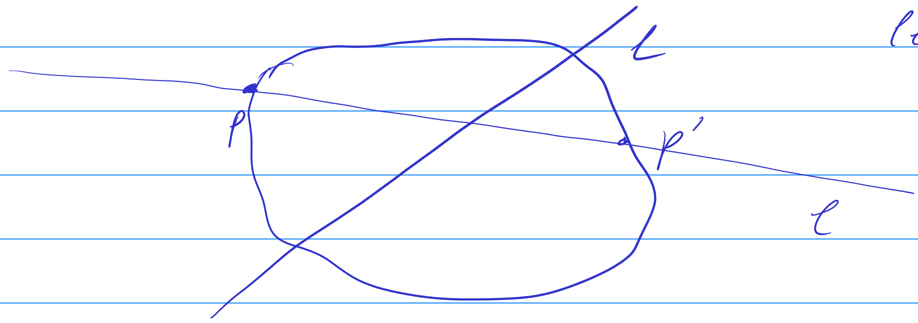


Rational parametrization of conics.

Let $f = x^2+y^2-z^2$.

The idea: Let X be a conic (solution to $f=0$)

Let $p \in X$ be some point, Let $\ell \subset \mathbb{P}^2$ be a line which does not contain p.
(a copy of $\mathbb{P}^1$)



for every $p' \in X, p' \neq p$ let ℓ' be the line passing through p, p', then $\ell \cap \ell'$ is some point of ℓ.

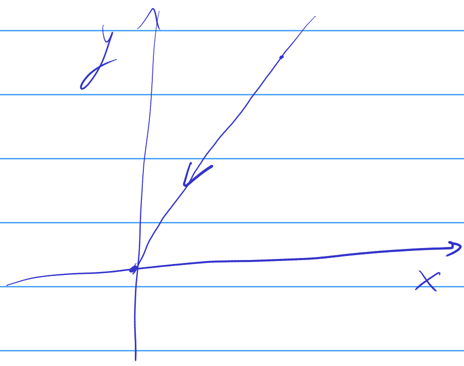
So we have a map $X \setminus \{p\} \to \ell$.

it turns out, it extends to an isomorphism.

afterwards:

$\deg f = 3 \rightsquigarrow$ elliptic curves.

One main question: understand all schemes of dimension 1.



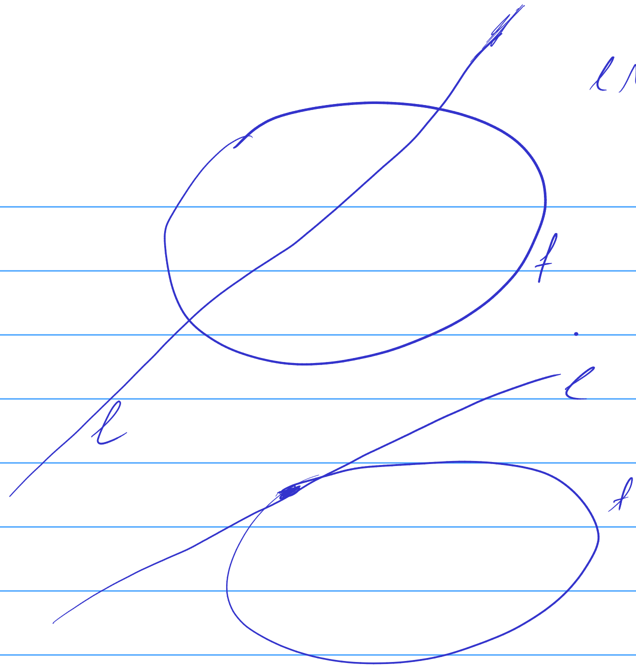
$y = kx$ $\quad k =$ "angle" $\quad$ in fact angle is $\arctan k$,

from $\quad$ k is called slope.

add $\frac{y}{x}$ to your ring

we want to also add the possibility $k = \infty$.

Question instead of m take (x, y^2) or m^2. do we get something new?



$\ell \cap f = * \cup *$

Scheme $= \operatorname{Spec}(R)$
R is a 2-dim vector space over k

$\deg f = 2$

$\operatorname{Spec} k[x]/x^2$

also 2-dim vector space over k.

just equations, sets of solutions
I $Z(I)$, Nullstellensatz

rings

$k \neq \bar{k}$, or want nilpotents in the ring.

gluing sets of solutions
$\mathbb{P}^n$

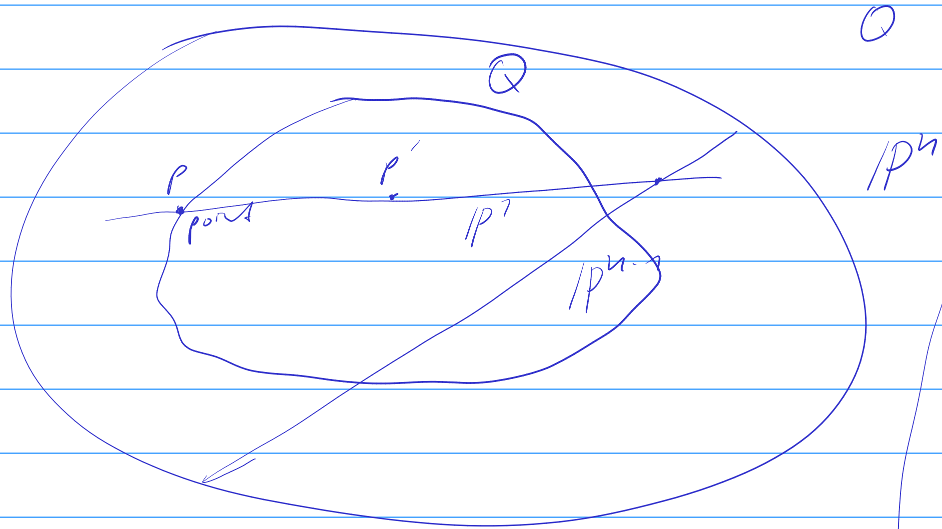
are called abstract varieties, over $k = \bar{k}$ the theory is equivalent to reduced schemes.

really important

$\exists$ notion of proper variety
replaces compact spaces in topology.

Some important spaces, like Grassmannian are these,
enumerative geometry needs proper variety.

Q degree $= 2$.



$\mathbb{P}^{n-1} \to Q$
probably the blow up of $p \in Q$.

$\mathbb{P}^n \setminus \{p\} \to \mathbb{P}^{n-1}$