$\mathbb{P}^n \times \mathbb{P}^m$ : $\mathbb{P}^n = \mathrm{Proj}\, k[x_0, \dots, x_n]$ $\mathbb{P}^m = \mathrm{Proj}\, k[y_0, \dots, y_m]$

Let $\underbrace{k[x_0, \dots, x_n]}_{= R} \otimes \underbrace{k[y_0, \dots, y_m]}_{= Q} = k[x_0, \dots, x_n, y_0, \dots, y_m]$ is bigraded $= \bigoplus_{i,j} R_i \otimes Q_j$

Let $S = \bigoplus R_i \otimes Q_i$. Then $\mathrm{Proj}(S) = \mathbb{P}^n \times \mathbb{P}^m$

To describe $\mathrm{Proj}(S) \subset$ projective space we look for generators of $S$. These are products $x_i y_j$ (form a basis of $S_1$, clearly generate $S$)

So we have $S$ = quotient of a polynomial ring in $(n+1)(m+1)$ variables. It turns out the ideal is generated by equations $(x_i y_j)(x_{i'} y_{j'}) = (x_i y_{j'})(x_{i'} y_j)$

denoting $t_{ij} = x_i y_j$ we have $t_{ij} t_{i'j'} = t_{ij'} t_{i'j}$.

$$S = k[(t_{ij})_{i=0}^{n}{}_{j=0}^{m}] / (t_{ij} t_{i'j'} - t_{ij'} t_{i'j})_{\substack{i,j,i',j' \\ j \neq j',\ i \neq i'}}$$

So $\mathbb{P}^n \times \mathbb{P}^n$ is a closed subscheme of $\mathbb{P}^{mn+m+n}$, for instance

<u>Example</u> $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$

generators: $t_{00}, t_{01}, t_{10}, t_{11}$

equations: $t_{00} t_{11} - t_{01} t_{10}$ $t_{01} t_{10} - t_{00} t_{11}, \dots$

all equations are the same,

so only one equation $t_{00} t_{11} - t_{01} t_{10}$.

By the way, any non-deg. quadratic form in 4 variables over $k = \bar{k}$ char $k \neq 2$ can be transformed into this form. So we have proved that non-degenerate quadrics in $\mathbb{P}^3$ are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

# Morphisms of schemes.

General philosophy: instead of looking at schemes, Sch we should fix some scheme $S$ called "base scheme", and look at pairs $\underset{\text{scheme}}{X}$, $\underset{\text{morphism}}{f: X \to S}$.

These pairs are called schemes over $S$. Morphisms of schemes$/S$ are commutative triangles:

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X' \arrow[d] \\ & S \end{tikzcd}$$

These way we obtain a category

$$Sch/_S.$$

Then every notion applied to schemes should be replaced by a notion of "scheme$/S$", so the notion will be applied to a morphism $f: X \to S$.

---

**Example** Notion = Scheme is affine

What is the corresponding notion for a morphism $f: X \to S$?

**Affine**

$X$ is affine if $X = \operatorname{Spec}(R)$.

**Affine morphism**:

Candidates:

1) $f: X \to S$ is affine if $S$ can be covered by affine opens $U_i$ so that $f^{-1}(U_i)$ is affine $(= \operatorname{Spec}(R_i))$

2) $\forall$ affine open $U \subset S$ we have $f^{-1}(U)$ is affine.

**Remark** any $X$ is a scheme over $\operatorname{Spec} \mathbb{Z}$. (Any ring $R$ has a unique homomorphism $\mathbb{Z} \to R$).

The notion for $X \to \operatorname{Spec} \mathbb{Z}$ should be equivalent to the notion for $X$.

Similarly, schemes over a field $k$ are just schemes over $\operatorname{Spec} k$, we expect the same.

2) $\Rightarrow$ 1)    It turns out 1) $\Rightarrow$ 2).

Properties are "good" if they are preserved under base change:

$f : X \to S$ affine

$\mathcal{O}(S) \to \mathcal{O}(X)$ invert some elements

$U \subset S$ is basic affine $\Rightarrow f^{-1}(U)$ is also affine

$$\begin{tikzcd} X \times_S T \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r] & S \end{tikzcd}$$

fiber product gives a morphism $X \times_S T \to T$. This is called the base change.

[We want if $X \to S$ satisfies the property, then $X \times_S T \to T$ does. (for all $T$, all $T \to S$).

For instance if $T \to S$ is just $T \subset S$ open then the base change is just the restriction $f^{-1}(T) \to T$.

The property "affine" is stable under the base change.

Why?

$$\begin{tikzcd} X \times_S T \arrow[r] \arrow[d, "\tilde{f}"] & X \arrow[d, "f"] \\ T \arrow[r, "\alpha"] & S \\ V_{ij} \arrow[u, phantom, "\subset" rotate=90] & U_i \arrow[u, phantom, "\subset" rotate=90] \end{tikzcd}$$

affine open cover of $\alpha^{-1}(U_i)$        affine open cover

$f^{-1}(U_i)$ is affine    what is $\tilde{f}^{-1}(V_{ij})$?

it has to be

$$V_{ij} \times_{U_i} f^{-1}(U_i).$$

This is fiber product of affine schemes over affine scheme, so it is affine!

$$= \operatorname{Spec}\left(\mathcal{O}(V_{ij}) \otimes_{\mathcal{O}(U_i)} \mathcal{O}(f^{-1}(U_i))\right).$$

Example

$$\begin{array}{c} X \\ \downarrow \\ \alpha: \operatorname{Spec} k \to S \end{array}$$

the fiber product as a set is simply the preimage of the point corresponding to $\alpha$.

If $f: X \to S$ is affine, then for every point $s \in S$ $\underline{f^{-1}(s)}$ is affine.

(by definition it is $\operatorname{Spec} k(s) \times_S X$.)

finite, finite type, closed.

Break ~ 10:44

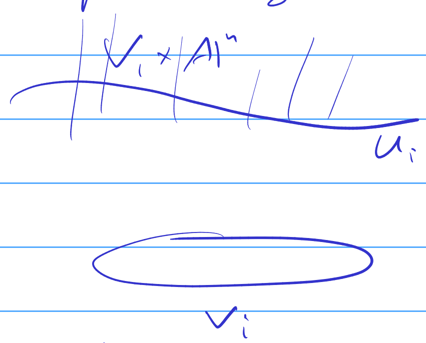
<u>Finite</u> <u>type</u>:

<u>Def</u> $f: X \to S$ is locally of finite type if we can cover $X$ by affine open sets $U_i$, and $S$ by $V_i$ so that $f(U_i) \subset V_i$, $\mathcal{O}(V_i) \to \mathcal{O}(U_i)$ is a finitely generated ring <u>extension</u>, i.e. $\mathcal{O}(U_i)$ is a quotient of $\mathcal{O}(V_i)[x_1, .., x_n]$.

Meaning: $U_i$ is represented by some equations in $V_i \times \mathbb{A}^n$

X

S




Nullstellensatz: for locally finite type image of a closed point is a closed point.

We say $f$ is of finite type if $\exists$ cover $V_i$ of $S$ so that $f^{-1}(V_i)$ is covered by finitely many affines of the form $\operatorname{Spec} \mathcal{O}(V_i)[x_1, .., x_n]$.

## Finite

( Finite type: finitely generated as a ring

finite: finitely generated as a module )

**Def** $f: X \to S$ is finite

if $\exists$ open affine cover $U_i$ of $S$ such that

1) $f^{-1}(U_i)$ is affine

2) $$\mathcal{O}(f^{-1}(U_i)) = \sum_{k=1}^{n} \mathcal{O}(U_i)\, x_k \qquad (x_1, \dots, x_k \in \mathcal{O}(f^{-1}(U_i)))$$

in other words

$U_i = \operatorname{Spec} R \qquad f^{-1}(U_i) = \operatorname{Spec} Q$

$$\begin{array}{c} f^{-1}(U_i) \\ \downarrow \\ U_i \quad R \end{array}$$

$R \to Q$

$Q$ is a module over $R$ and the condition is that $Q$ is a finitely generated module.

---

**Geometrically** Suppose $f$ is finite (finite type)

$s \in S$ point then $f^{-1}(s)$ finite over $\operatorname{Spec} k(s)$ (finite type over $\operatorname{Spec} k(s)$)

So $f^{-1}(s) = \operatorname{Spec} Q_s$
$Q_s$ is finite dimensional as a vector space; in particular $f^{-1}(s)$ has finitely many points.

$f^{-1}(s)$ is glued from finitely many affine subschemes of $\mathbb{A}^n$.

$R$ ring, finite dimensional v. sp. $/ k$

$R/\mathrm{Nil}(R) \cong \prod k_i$. $k_i$ is a field extension of $k$.

$\mathfrak{p}_i$ minimal prime $\quad R/\mathfrak{p}_i$ is a domain, finite dim $/k$. So is a field.

$R/\mathrm{Nil}(R) \to \prod R/\mathfrak{p}_i$ is iso.

## Examples

$$\begin{tikzcd}
\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1 \arrow[d, "z \mapsto z^2"'] & \mathbb{A}^1 \setminus \{0\} \arrow[l] \arrow[l, bend right, "z \mapsto z^{-1}"'] \\
\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1 &
\end{tikzcd}$$

$$(z^{-1})^2 = (z^2)^{-1}$$

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[d] & z \arrow[d] \\
\mathbb{P}^1 & z^2
\end{tikzcd}$$

When I write $z \to z^2$ $\mathbb{A}^1 \to \mathbb{A}^1$ I don't mean that $z = 2$ goes to $z^2 = 4$, $z = \sqrt{2}$ goes to $z^2 = 2$.

I mean that there is a ring homomorphism.

**More carefully**

$\mathbb{A}^1$ coordinate $z$ $\qquad$ $\mathbb{A}^1$ coordinate $y$

$z \to z^2$ means $y = z^2$ defines a homomorphism

$$k[y] \to k[z]$$
$$y \to z^2$$

$$k[z] \xleftarrow{y \mapsto z^2} k[y].$$

So we have a map $\mathbb{P}^1 \to \mathbb{P}^1$.

Claim: This is finite. ($\mathbb{P}^1$ is not affine, but the morphism is)

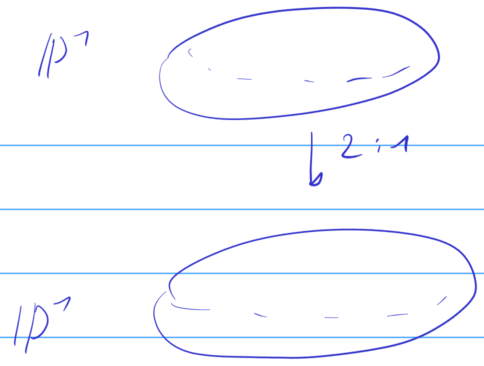
Look: over $\mathbb{A}^1 \subset \mathbb{P}^1$, $\mathbb{A}^1 = \operatorname{Spec} k[y]$

$$k[y] \xrightarrow{y = z^2} k[z]$$

$$k[z] \cong k[z^2] \oplus z\,k[z^2] = k[y] \oplus z\,k[y]$$

so is a finitely generated module.

Geometrically




preimage of every point $\neq 0, \infty$ is 2 points,

preimage of 0 is $: z^2 = 0$

Spec $k[z]/z^2$ "fat point".

$\boxed{\text{Ramified coverings of Riemann surfaces } \xleftrightarrow{\text{bijection}} \text{ finite maps of algebraic curves.}}$

Break ~ 11:34

Not-finite morphism:

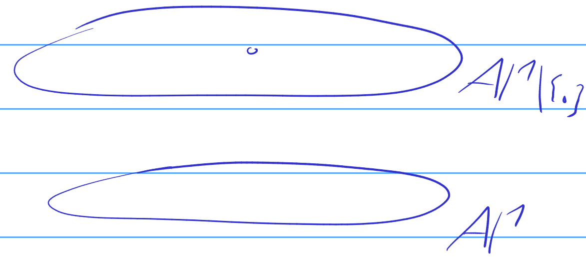
$$\mathbb{A}^1 \setminus \{0\} \longrightarrow \mathbb{A}^1$$
$$\text{Spec } k[t, t^{-1}] \qquad \text{Spec } k[y]$$

$t = y$

$$\underset{R}{k[y]} \longrightarrow \underset{Q}{k[t, t^{-1}]} \quad \text{not finite:}$$

take finitely many elements $f_1, \dots, f_n \in Q$, suppose $N^{>>0}$ is such that $f_1, \dots, f_n$ contain only $t^k$ $k > -N$ then $t^{-N} \notin Rf_1 + \dots + Rf_n$, we cannot have $Q = Rf_1 + \dots + Rf_n$.

Geometrically




preimage of every point is finitely many points.

**Question** What's the difference ?

$f: X \to S$ finite $\qquad$ $\forall s \in S$ $f^{-1}(s)$ has finitely many points

**Remark** any homo. of rings $\varphi: R \to Q$ can be uniquely decomposed

$$\underset{\text{surjective}}{R \to} R/\mathrm{Ker}\,\varphi \underset{\text{injective.}}{\to} Q$$

Geometrically $\quad \mathrm{Spec}\, Q \to \underbrace{\mathrm{Spec}(R/\mathrm{Ker}\,\varphi) \to \mathrm{Spec}\, R}_{\text{closed subscheme}}$

**Generally** surjective homomorphisms of rings $\longleftrightarrow$ closed embeddings.

What about $R \to Q$ injective ? $\iff$ image of $X$ is dense in $S$.

$\varphi^*: \underset{X}{\mathrm{Spec}\, Q} \to \underset{S}{\mathrm{Spec}\, R}$

$\boxed{\text{assuming } S \text{ is reduced}}$.

$$\Updownarrow$$

$$\overline{\varphi^*(X)} = S \iff$$

$$Z(I(\varphi^*(X))) = S = Z(\{0\})$$

$$\Updownarrow$$

$$I(\varphi^*(X)) = \mathrm{rad}\{0\}$$

$$\Updownarrow$$

$$\{f:\ \varphi(f) \in \mathrm{rad}\{0\}\}$$

$$\varphi^{-1}\,\mathrm{rad}\{0\} = \mathrm{rad}\{0\}$$

If $\overline{\varphi^*(X)} = S$ we say $\varphi^*$ is **dominant**.

Conclusion

$$\mathrm{Spec}\, Q \underset{\text{dominant}}{\to} \mathrm{Spec}(R/\mathrm{Ker}) \underset{\text{closed embedding}}{\to} \mathrm{Spec}(R)$$

$S$ reduced $\Rightarrow$ $\varphi^{-1}(\{0\}) \subset \varphi^{-1}(\mathrm{rad}\{0\}) = \mathrm{rad}\{0\} = \{0\}$

$\overline{\varphi^*(X)} = S \Rightarrow \varphi$ injective.

if $Q$ is reduced $\Rightarrow$

$R/\mathrm{Ker}\,\varphi$ is reduced.

More generally

$f: X \to S$ we have $\overline{f(X)} \subset S$ closed subscheme

we can decompose (if $X$ is reduced)

$f: X \to \overline{f(X)} \to S$

dominant closed embedding.

The map $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ was dominant.

<u>Def</u> A map is closed $f: X \to S$ if $\forall Z \subset X$ closed $f(Z)$ is also closed.

in particular, $f(X)$ is closed, so in the decomposition $X \to \overline{f(X)} \to S$ the dominant map $X \to \overline{f(X)}$ is in fact surjective.

Similarly to: if $X$ is compact, then $f$ is closed.

<u>Theorem</u> $f: X \to S$ is finite if and only if it is affine, of finite type and universally closed.

<u>universally closed</u>: $\forall T \to S$ $T \times_S X \to T$ is closed.

<u>Discussion</u> finite $\Rightarrow$ closed

we have seen $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ not closed! so is not finite

if we prove finite $\Rightarrow$ closed, we have

finite $\Rightarrow$ universally closed. (stable under base change)

So we have the Theorem in one direction.

<u>Exercise</u> prove the Theorem.

What about all $f^{-1}(s)$ have finitely many points?

Claim $f: X \to S$ $\dim X \leq \dim S + \max_{s \in S \text{ closed}} \dim f^{-1}(s)$ if $f$ is finite type.

if $f^{-1}(s)$ is finite $\Rightarrow$ $\dim X \leq \dim S$.

Ex $\quad k[x]_{(x)}$ is not finitely generated, it has 2 points, but $\dim \neq 0$.

finite type is important.

$\operatorname{Spec} R \to \mathbb{P}^1$

$\operatorname{Spec} R = U_1 \cup U_2 \qquad U_1 \to \mathbb{A}^1 \rightsquigarrow \frac{x}{f^m}$

$U_2 \to \mathbb{A}^1 \rightsquigarrow \frac{y}{g^m}$

$U_1 \cap U_2 \to \mathbb{A}^1 \setminus \{0\}$

$$\begin{tikzcd} & U_1 \arrow[r] & \mathbb{A}^1 \\ U_1 \cap U_2 \arrow[ur] \arrow[r] \arrow[dr] \arrow[d] & \mathbb{A}^1 \setminus \{0\} \arrow[ur] & \\ & \mathbb{A}^1 \setminus \{0\} \arrow[u, "z \to z^{-1}"'] \arrow[d] & \\ U_2 \arrow[r] & \mathbb{A}^1 & \end{tikzcd}$$

$U_1 = U_f$

$U_2 = U_g \qquad f\alpha + g\beta = 1$

$\frac{xy}{(fg)^m} = 1$ in $R[f^{-1}, g^{-1}]$

changing $m, f, g \to$ assume $f\alpha + g\beta = 1$

$\frac{x}{f} \quad \frac{y}{g} \qquad xy = fg$

$f \to f\alpha \qquad x \to x\alpha$

$g \to g\beta \qquad y \to y\beta$

$$\boxed{\begin{array}{l} f + g = 1 \\ xy = fg \end{array}}$$

$\begin{pmatrix} f & x \\ y & g \end{pmatrix} \quad \det = 0, \ \operatorname{tr} = 1$

Next Friday:

1) Correspondence between $\mathrm{Mor}(X, \operatorname{Spec} R)$ and

$$\mathrm{Hom}(R, \mathcal{O}_X(X)) \\ \shortparallel \\ \Gamma(\mathcal{O}_X, X)$$

2) Maps to $\mathbb{P}^1$, $\mathbb{P}^n$

3) Blow up of say $\mathbb{P}^2$ (Projective schemes)

4) Exam topics recap