LA Evaluation

Philosophy

functions $f \in \mathcal{O}_X(U)$

notion of ringed space

?

points $x \in X$ $\longleftrightarrow$ open sets $U \subset X$

T_0 space

points are determined by open sets that contain them

$x = y \in X \iff \forall U \quad x \in U \iff y \in U$

Statement

$\mathrm{Mor}(X, \mathrm{Spec}\, B) \simeq \mathrm{Hom}(B, \mathcal{O}_X(X))$

Given $\varphi : X \to \mathrm{Spec}\, B$ a morphism of ringed spaces + assumption we have a ring homomorphism $B \to \mathcal{O}_X(X)$ (because it's a morphism of ringed spaces)

So we have a map

$K : \mathrm{Mor}(X, \mathrm{Spec}(B)) \to \mathrm{Hom}(B, \mathcal{O}_X(X))$

Claim this is a bijection.

Construction $L : \mathrm{Hom}(B, \mathcal{O}_X(X)) \to \mathrm{Mor}(X, \mathrm{Spec}(B))$

$KL = \mathrm{Id}$:

Let $\psi : B \to \mathcal{O}_X(X)$ $\quad x \in X$

To construct $\varphi : X \to \mathrm{Spec}(B)$

1) On points $\varphi(x) = y$ such that

$\forall f \in B \quad f(y) \neq 0 \iff \psi(f)(x) \neq 0$.

$\psi(f) = f \circ \varphi$

We have a notion of function vanishing at a point

($f(x) \neq 0$ if $\exists U \ni x$ s.t. $f|_U$ is invertible)

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & \mathrm{Spec}\, B = Y & \xrightarrow{f} & \mathbb{A}^1 \\ \Psi & & \Psi & & \\ x & & y = \varphi(x) & & \end{array}$$

2) prove φ continuous

3) construct homomorphisms of rings $\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$.

$$\mathrm{Mor}(X, \mathrm{Spec}\, B) \underset{K}{\overset{L}{\rightleftarrows}} \mathrm{Hom}(B, \mathcal{O}_X(X))$$

$\circlearrowleft$ identity.

$\Rightarrow$ K is surjective

we need to show: K is injective.

or LK = identity

so let $\varphi : X \to \mathrm{Spec}\, B$ $\qquad \varphi' = LK(\varphi))$

we need to show: $\varphi = \varphi'$.

(1) show $\forall x \quad \varphi(x) = \varphi'(x)$.

suppose $\varphi(x) = y$, we need:

$$\forall f \in B \qquad f(y) \neq 0 \iff (\varphi^* f)(x) \neq 0.$$

Claim The following are equivalent for a morphism of ringed spaces $\varphi : X \to Y$, where X, Y are schemes.

1) $\forall U \subset Y, \ \forall f \in \mathcal{O}_Y(U) \quad \forall x \in \varphi^{-1}(U)$
$f(\varphi(x)) \neq 0 \iff (\varphi^* f)(x) \neq 0$

2) $\forall x \in X$ the map of local rings $\mathcal{O}_{X,x} \to \mathcal{O}_{Y, \varphi(x)}$ is a local ring map

standard definition of map of locally ringed spaces

3) $\forall x \in X$, $U \ni x$ affine $V \ni f(x)$ s.t. $f(U) \subset V$

$U \longrightarrow V$

$x \qquad f(x)$

the prime ideal $\subset \mathcal{O}_Y(V)$ corresponding to $f(x)$ is the preimage of the prime ideal $\subset \mathcal{O}_X(U)$ corresponding to x via the map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$.

just says that the bijection between points and prime ideals is compatible with the map

After we have $\varphi(x) = \varphi'(x)$, the equality of the homomorphisms on function rings is evident.

\begin{tikzcd}
\mathcal{O}_X(X) \arrow[d] & B = \mathcal{O}_Y(Y) \arrow[l, "\varphi = \kappa(\psi)"'] \arrow[d, "r"] \\
\mathcal{O}_X(f^{-1}(U)) & \mathcal{O}_Y(U) \arrow[l, "\varphi_U, \varphi_U'"]
\end{tikzcd}

$\varphi_U \circ r = \varphi_U' \circ r$, and since r is a localization $\varphi_U = \varphi_U'$.

So the proof is complete modulo the claim.

$\varphi(U) \subset V$

$$\mathcal{O}_Y(V) \xrightarrow{\varphi_V} \mathcal{O}_X(f^{-1}(V)) \xrightarrow{res} \mathcal{O}_X(U)$$

More on L:

$\psi : B \to \mathcal{O}_X(X)$

constructed $\varphi = L(\psi)$ on points, $\operatorname{Spec} B$

want to define φ_V $(V \subset Y)$

need φ_V:

\begin{tikzcd}
\mathcal{O}_X(X) \arrow[d] & B = \mathcal{O}_Y(Y) \arrow[l, "\psi"'] \arrow[d] \\
\mathcal{O}_X(\varphi^{-1}(V)) & \mathcal{O}_Y(V) \arrow[l, dashed, "\varphi_V"]
\end{tikzcd}

by universal prop. of localization $\exists!\, \varphi_V$.

Next question. Blow ups

$\{xy = zw\} \subset \mathbb{P}^3$

Blow up Q in $p = (0,0,0,1)$:

cover $\mathbb{P}^3$ by 4 charts U_x, U_y, U_z, U_w

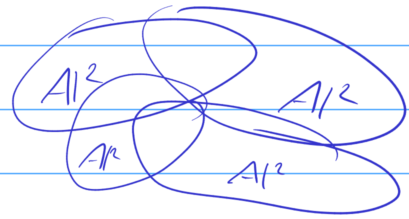
in each chart we have:

Say take U_x:

$$\left[k[x,y,z,w,x^{-1}]/(xy-zw)\right]_0 = k\left[\frac{y}{x}, \frac{z}{x}, \frac{w}{x}\right] \Big/ \left(\frac{y}{x} - \frac{z}{x}\cdot\frac{w}{x}\right), \text{ so}$$

$U_x = \mathbb{A}^2$ with coords $\frac{z}{x}, \frac{w}{x}$

Similarly U_y, U_z, U_w.



what about p?

$p \notin U_x, U_y, U_z$

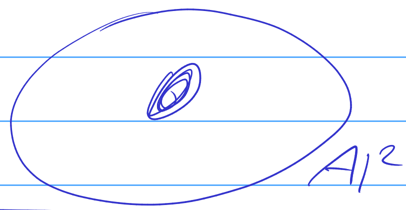
$p \in U_w$

U_w has coord's $\frac{x}{w}, \frac{y}{w}$, so p is the origin in $U_w = \mathbb{A}^2$.

$\mathrm{Blowup}_p\, Q$ = glueing together blowups of U_x, U_y, U_z, U_w at p. (for U_x, U_y, U_z p is not there, do nothing)

only blow up U_w at p

map between glueing $U_x \cup U_y \cup U_z$ and U_w does not contain p.



$U_x \cap U_w = ?$

u_1, v_1 u_2, v_2

$u_1 = z/x$ $u_2 = \frac{x}{w}$

$v_1 = w/x$ $v_2 = \frac{y}{w}$

$$\boxed{\begin{aligned} u_2 &= \frac{1}{v_1} \\ v_2 &= u_1 \end{aligned}}$$

On U_z:

u_3, v_3

$u_3 = \frac{1}{v_2}$

$v_3 = u_2$

$$\begin{tikzcd} U_x \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & U_w \arrow[d, leftrightarrow] \\ U_z \arrow[r, leftrightarrow] & U_y \end{tikzcd}$$

Blow up U_w:

union of U_w^1 & U_w^2 $\qquad U_z \cap U_w = U_z \cap U_x \cap U_y$.

$\uparrow$ u_1, s $\qquad$ $\uparrow$ v_1, t

$v_1 = u_1 s$ $\qquad$ result: 5 copies of $\mathbb{A}^2$ glued together

$t = \frac{1}{s}$

$\mathbb{P}^2$, 2 points

U_a, U_b, U_c $\qquad$ 3 copies of $\mathbb{A}^2$ glued together

blow up $\qquad q_1 = (1,0,0) \qquad q_2 = (0,1,0)$

q_1 is contained only in U_a

q_2 —— " —— U_b,

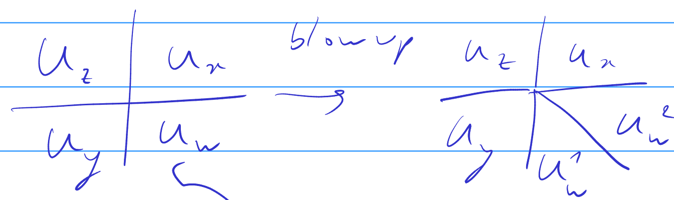
So we can blow up q_1 in $U_a \to U_a^1, U_a^2$

q_2 in $U_b \to U_b^1, U_b^2$

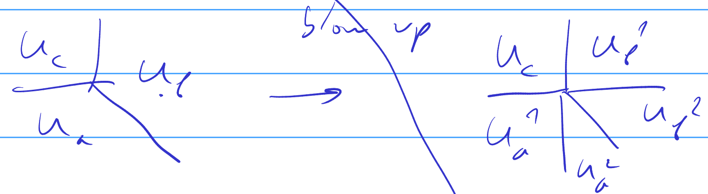
in total we obtain

5 copies of $\mathbb{A}^2$: $U_a^1, U_a^2, U_b^1, U_b^2, U_c$

Graphically $\quad Q$



$\mathbb{P}^2$



Combinatorially looks similar,

toric geometry allows to encode gluing maps by this diagrams

$\frac{d u_x}{u_w}$ matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ corresponds to the change of variables

$$u_2 = \frac{1}{v_1} = v_1^{(-1)} u_1^{(0)}$$

$$v_2 = u_1 = v_1^{(0)} u_1^{(1)}$$

Powerful stuff. (toric geometry).

Maps to $\mathbb{P}^1$

How to describe all maps from $X = \operatorname{Spec} A$ to $\mathbb{P}^1$, A arbitrary ring?

$\mathbb{P}^1 = U_x \cup U_y$ we can describe a map $f: X \to \mathbb{P}^1$ by covering X by $f^{-1}(U_x)$ $f^{-1}(U_y)$, and describing the restrictions of f to them.

<u>Lemma</u> if $X = \operatorname{Spec} A$ is covered by open sets $U_1, \dots, U_n$, then there exist basic open sets $V_1, \dots, V_n$ such that $V_i \subset U_i$, $\bigcup_{i=1}^{n} V_i = X$.

<u>Pf</u>

Cover U_i by basic sets U_{ij}, i.e. $U_i = U_{i1} \cup U_{i2} \dots U_{im_i}$

Then all $\{U_{ij}\}$ cover X

Suppose $U_{ij} = U_{f_{ij}}$, some $f_{ij} \in A$

($Z_1, Z_2 \subset X$ closed $Z_1 \cap Z_2 = \emptyset \Rightarrow \exists f$ s.t. $f|_{Z_1} = 0$ $f|_{Z_2} = 1$)

$\Rightarrow$! (f_{ij}) generate unit ideal (1).

$$1 = \sum c_{ij} f_{ij} \qquad c_{ij} \in A.$$

Let $g_i = c_{i1} f_{i1} + c_{i2} f_{i2} + \dots + c_{im_i} f_{im_i}$.

Then $1 = \sum g_i$, so $V_i = U_{g_i}$ cover X

$V_i \subset U_i$ because $g_i \in (f_{i1}, \dots, f_{im_i})$. ∎

$X \setminus V_i \supset X \setminus U_i = \bigcap_j X \setminus U_{ij}$

$\Uparrow$

$(g_i) \subset (f_{i1}, \dots, f_{im_i})$.

Applying it to $f^{-1}(U_x)$, $f^{-1}(U_y)$ we obtain basic open sets $V_1, V_2 \subset X$, $f(V_1) \subset U_x$ $f(V_2) \subset U_y$.

So suppose $V_1 = U_a$ $V_2 = U_b$. $\quad a, b \in A$

$V_1 \to U_x \cong \mathbb{A}^1$ is given by c, $V_2 \to U_y \cong \mathbb{A}^1$ by d.

Obtain $a, b \in A$ $\quad c = a^{-k} c_1 \quad d = b^{-k'} d_1$

on $V_1 \cap V_2 = U_{ab}$

$c \cdot d = 1 \iff$

$$\cancel{\frac{c_1}{a^k} - \frac{d_1}{b^{k'}} = \frac{c_1 b^{k'} - a^k d_1}{a^k b^{k'}}} \quad \exists\, m \quad (c_1 b^{k'} - a^k d_1)(ab)^m = 0$$

can assume $a + b = 1$. $\quad$ increasing k, k', m, replacing a, b by their powers

can assume $\quad c = \frac{c_1}{a}. \quad d = \frac{d_1}{b}, \quad c_1 d_1 = ab$

$a + b = 1$

$\alpha a + \beta b = 1$

$a \to \alpha a$

$b \to \beta b$

$$\to \quad M = \begin{pmatrix} a & c_1 \\ d_1 & b \end{pmatrix} \quad \text{trace} = 1, \ \det = 0.$$

$$M^2 = M.$$

Break ~ 11:34.

If $a \in A$ $\quad U_a \subset \operatorname{Spec} X$

functions on U_a by definition are elements of the ring $S^{-1}A$ $\quad S = \{1, a, a^2, \ldots\}$

elements of $S^{-1}A$ are fractions $\frac{x}{a^n}$.

Summary of topics covered

1) Commutative algebra

Lectures 1–6

rings, ideals, fractions, quotient ring,
their universal properties (equivalently, representing a functor).

prime ideals, maximal ideals, radical, Spec,
Zariski topology, Zero set of ideal, $I(Z)$ ideal of a set.
$Z(I)$

relationship with radical, operations ($\cup$ $\cap$).

Nullstellensatz

Noetherian rings

minimal primes

dimension theory: Krull dimension of a top. space,
height of a prime ideal,
relationship height $\longleftrightarrow$ # of generators of ideal
Krull Hauptidealsatz.

computing dimension (say, for polynomial ring).

Local rings, Nakayama, Artinian rings.

2 Schemes Ringed spaces

Construction of Spec as a ringed space

Gluing schemes from affine schemes

Morphisms of schemes

Affinization

Proj (by gluing, direct construction as a ringed space)

Functor of points (what it means to represent a functor)

The idea of Fiber products.

The idea of Finite type, Finite morphisms.

> Examples

polynomial rings,
projective space,
blow up
conics on $\mathbb{P}^2$
Some schemes with finitely many points,
like $k[x]/(x^2)$.

Complicated

generic points of subvarieties $\subset$

Next semester:

With a goal in mind to get to
GIT and moduli spaces,
we study some chapters in Hartshorne, Chap II
Mumford's book on GIT,
$\boxed{\text{curves (Chap IV)}}$

see grt-home.org Seminars,
videos of talks
(use search)