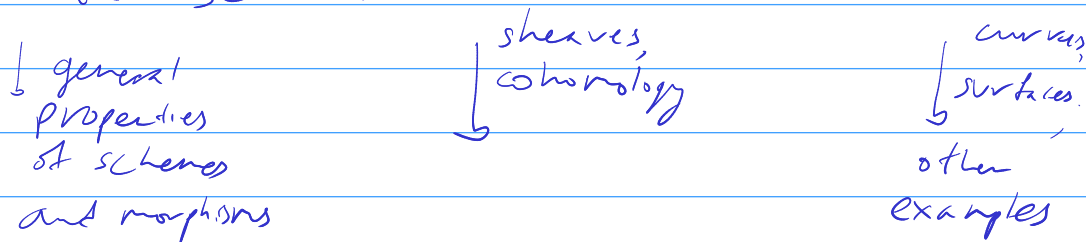


Algebraic geometry

Basic rules:

- 1) Use cameras if you can!
- 2) Please interrupt ^{and} use voice.
(I don't always see chat)

Plan for the semester



References see vtrnd

the best is The Stacks Project (online)

Schedule

2h Friday

1h lecture + 1h PS on Monday

↑
examples, sometimes homework.

Today: Abelian categories.

Basic idea: We have categories "like" the category of vector spaces,

there is some toolbox to work with them.

Main example fix a ring A (unital, commutative),

consider the category of A -modules.

Later, for a scheme X we will have the category of coherent quasi-coherent sheaves.

Def Category = Objects $X \in Ob$
 Morphisms $f: X \rightarrow Y \in \text{Mor}(X, Y)$
 \parallel
 $\text{Hom}(X, Y)$

Operations:

Composition $f: X \rightarrow Y, g: Y \rightarrow Z, h: Z \rightarrow W$
 $\exists Id_X \quad \forall X \in Ob$

Axioms
 $f \circ Id_X = Id_Y \circ f \quad \forall f: X \rightarrow Y$
 $(f \circ g) \circ h = f \circ (g \circ h) \quad h: X \rightarrow Y, g: Y \rightarrow Z, f: Z \rightarrow W$

Def A linear category is a category in which $\text{Hom}(X, Y)$ is an abelian group.

If A is a ring, an A -linear category \mathcal{C} is such that $\text{Hom}(X, Y)$ is an A -module.

such that the composition $f, g \rightarrow f \circ g$ is linear (A -linear) in f and in g .

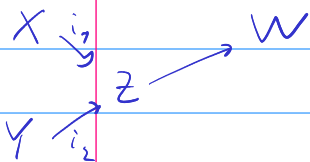
Def A linear (A -linear) category is additive if finite direct sums exist. (& zero object exists)

What does it mean?

$X, Y \in Ob(\mathcal{C})$

Z is a direct sum of X, Y

if 1) we have $i_1: X \rightarrow Z$
 $i_2: Y \rightarrow Z$



2) $\forall W$

$\text{Hom}(Z, W) \rightarrow \text{Hom}(X, W) \times \text{Hom}(Y, W)$

is a bijection.

(By 2) I know $\text{Hom}(Z, W)$ for any W . By Yoneda lemma Z is completely determined up to isomorphism.

Def An additive category is abelian if kernels & cokernels exist, and $f: X \rightarrow Y$
 $\text{Ker}(\text{Coker}(f)) = \text{Coker}(\text{Ker}(f))$.

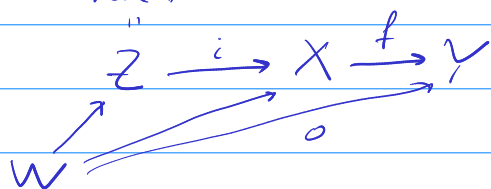
Explanation

What is a kernel?

given $f: X \rightarrow Y$ a kernel of f is Z , together with $i: Z \rightarrow X$ such that $\forall W$
 $\text{Hom}(W, Z) \rightarrow \text{Hom}(W, X)$ is injective and the image consists of precisely those $\text{Hom}(W, X)$ which are zero after composing with f .

Thinking:

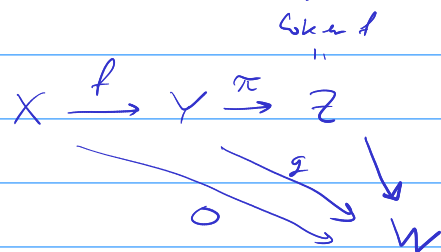
$f: X \rightarrow Y$
 $\text{Ker}(f) \subset X \quad \text{Ker}(f) \rightarrow X$
 $\text{Hom}(W, \text{Ker}(f)) = \{g: \text{Hom}(W, X) \mid f \circ g = 0\}$



What is a cokernel? (invert all arrows!)

$X \xrightarrow{f} Y$

$\text{Coker}(f)$ is a object Z and a map $\pi: Y \rightarrow Z$, such that



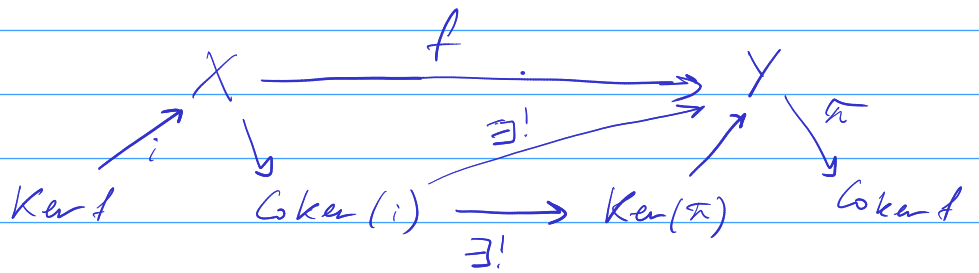
$\text{Hom}(Z, W) \rightarrow \text{Hom}(Y, W)$ is injective and its image consists of those $g \in \text{Hom}(Y, W)$ which satisfy $g \circ f = 0$.

In linear algebra g vanishes on $\text{Im } f \Rightarrow g$ a map from $Y/\text{Im } f$. So, $\text{Coker}(f) = Y/\text{Im } f$.

By Yoneda lemma if ker/coker exists, it is unique up to isomorphism.

Remark All these definitions can be reformulated using the idea of universal property, and their uniqueness can be proved using univ. property methods.

Finally, what does $\text{Ker}(\text{Coker})$, $\text{Coker}(\text{Ker})$ have to do with each other?



$$\text{Coker}(i) \xrightarrow{\exists!} \text{Ker}(\pi) \xrightarrow{\exists!} \text{Coker}(f)$$

$X \rightarrow \text{Coker}(i) \rightarrow Y \rightarrow \text{Coker}(f)$
 $\quad \quad \quad \cap 0$
 $\quad \quad \quad = \pi \circ f = 0.$

The assumption is: $\text{Coker}(i) \rightarrow \text{Ker}(\pi)$ is an isomorphism.

PS

Exercise 1

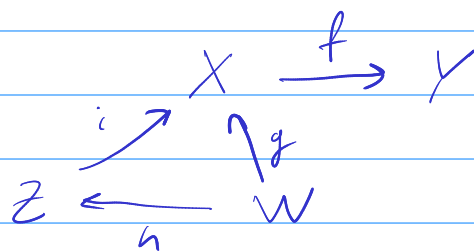
Formulate the definition of $\text{Ker}(f)$ via universal property and prove that it exists, it is unique up to iso.

5 min ~ 10:43

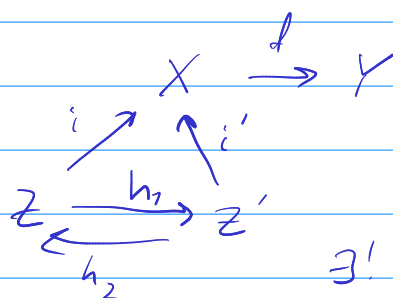
Ex 2 Give an

example of a non-abelian additive category.

Universal property description.

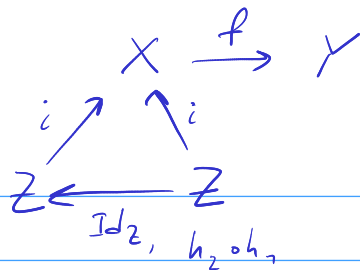


- 1) $f \circ i = 0$
- 2) $\forall g: W \rightarrow X$
s.t. $f \circ g = 0$
 $\exists! h: W \rightarrow Z$ s.t. $g = i \circ h.$



- 1) $f \circ i = 0$
- $\exists! h_1$ s.t. $i = i' \circ h_1$
- $\exists! h_2$ s.t. $i' = i \circ h_2$

$$i = i \circ h_2 \circ h_1$$



$\text{Id}_Z, h_2 \circ h_1$ both make the diagram commutative.

Simon

Similarly $h_1 \circ h_2 = \text{Id}_Z \Rightarrow Z$ is iso to Z' .
by the uniquenesses $\text{Id}_Z = h_2 \circ h_1$.

Exercise 2

Non-abelian, additive category.

one (probably) abelian

Banach spaces? $X \xrightarrow{f} Y$ $\text{Ker } f$ is a Banach space
 $\text{Coker } f = Y / \overline{\text{Im } f}$ will work.

Topological modules / top ring

take vector (k a field) remove 1-dim vector spaces $\therefore C$

Take $X \xrightarrow{f} Y$ $\dim X, Y > 1$
 $\dim \text{Ker } f = 1$ (in the usual sense).

want to show

there is no $\text{ker}(f) \in C$.

$\text{Ker}(f) \rightarrow \mathbb{R}^2$ injective $\Rightarrow \dim(\text{Ker } f) = 0$
 $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2 \rightarrow \mathbb{R}^2$
 $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \searrow \uparrow \text{Ker}(f)$
 $f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
 $\text{ker } f = 0$

contradiction.

Homework Find a category which is additive, ker, Coker exist, but the assumption $\text{Ker}(\text{Coker}) = \text{Coker}(\text{Ker})$ fails.