Algebraic geometry

Basic rules:

1) Use cameras if you can!

2) Please interrupt, and use voice.
(I don't always see chat)

<u>Plan</u> for the semester

- general properties of schemes and morphisms
- sheaves, cohomology
- curves, surfaces, other examples

References see Ufind
the best is The Stacks Project (online)

Schedule
2h Friday
1h lecture + 1h PS on Monday
examples, sometimes homework.

Today: Abelian categories.

Basic idea: We have categories "like" the category of vector spaces, there is some toolbox to work with them.

<u>Main example</u> fix a ring A (unital, commutative), consider the category of A-modules.

Later, for a scheme X we will have the category of coherent quasi-coherent sheaves.

Def Category = Objects $X \ni Ob$

Morphisms $f: X \to Y \ni Mor(X,Y)$
$$\| $$
$Hom(X,Y)$

Operations:

Composition $f: X \to Y \quad g: Y \to Z \quad g \circ f: X \to Z$

$\exists Id_X \quad \forall X \in Ob$

Axioms $\quad f \circ Id_X = Id_Y \circ f \quad \forall f: X \to Y$

$(f \circ g) \circ h = f \circ (g \circ h) \quad h: X \to Y,\ g: Y \to Z,\ f: Z \to W$

Def A linear category is a category in which $Hom(X,Y)$ is an abelian group.

If A is a ring, an A-linear category —"— $Hom(X,Y)$ is an A-module.

such that the composition $f, g \to f \circ g$ is linear (A-linear) in f and in g.

Def A linear (A-linear) category C is additive if finite direct sums exist. (& zero object exists)

What does it mean?

$X, Y \in Ob(C)$

Z is a direct sum of X, Y

if 1) we have $i_1: X \to Z$

$i_2: Y \to Z$

$$\begin{array}{l} X \xrightarrow{i_1} Z \to W \\ Y \xrightarrow{i_2} Z \end{array}$$

2) $\forall W$

$Hom(Z,W) \to Hom(X,W) \times Hom(Y,W)$

is a bijection.

(By 2) I know $Hom(Z,W)$ for any W. By Yoneda lemma Z is completely determined).

up to isomorphism.

<u>Def</u> An additive category is <u>abelian</u> if kernels & cokernels exist, and $f: X \to Y$

$\mathrm{Ker}(\mathrm{Coker}(f)) \cong \mathrm{Coker}(\mathrm{Ker}(f))$.

<u>Explanation</u>

Thinking:

$f: X \to Y$

$\mathrm{Ker}(f) \subset X$ $\quad \mathrm{Ker}(f) \to X$

$\mathrm{Hom}(W, \mathrm{Ker}(f)) =$
$\{g : \mathrm{Hom}(W, X) \mid f \circ g = 0\}$

What is a kernel?

given $f: X \to Y$ a kernel of f is Z, together with $i: Z \to X$ such that $\forall W$ $\mathrm{Hom}(W, Z) \to \mathrm{Hom}(W, X)$ is injective and the image consists of precisely those $\mathrm{Hom}(W, X)$ which are zero after composing with f.

$$\begin{tikzcd}
& \overset{\text{"Ker } f\text{"}}{Z} \arrow[r, "i"] & X \arrow[r, "f"] & Y \\
W \arrow[ur] \arrow[urr] \arrow[urrr, "0"'] & & &
\end{tikzcd}$$

What is a cokernel? (invert all arrows!)

$X \xrightarrow{f} Y$

$\mathrm{Coker}(f)$ is a object Z and a map $\pi: Y \to Z$, such that

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[drrr, "0"'] & Y \arrow[r, "\pi"] \arrow[drr, "g"] & \overset{\text{Coker } f}{Z} \arrow[dr] & \\
& & & W
\end{tikzcd}$$

$\mathrm{Hom}(Z, W) \to \mathrm{Hom}(Y, W)$ is injective and its image consists of those $g \in \mathrm{Hom}(Z, W)$, which satisfy $g \circ f = 0$.

In linear algebra g vanishes on $\mathrm{Im} f \Rightarrow g$ a map from $Y/\mathrm{Im} f$. So, $\mathrm{Coker}(f) = Y/\mathrm{Im} f$.

By Yoneda lemma if ker/coker exists, it is unique up to isomorphism.

<u>Remark</u> All these definitions can be reformulated using the idea of universal property, and then uniqueness can be proved using univ. property methods.

Finally, what does Ker(Coker), Coker(Ker) have to do with each other?

$$\begin{tikzcd} & X \arrow[rrr, "f"] \arrow[dr] & & & Y \arrow[dr, "\pi"] & \\ \mathrm{Ker} f \arrow[ur, "i"] & & \mathrm{Coker}(i) \arrow[r, "\exists!"'] \arrow[urr, "\exists!"] & \mathrm{Ker}(\pi) \arrow[ur] & & \mathrm{Coker} f \end{tikzcd}$$

$f \circ i$ (!) $\quad \pi \circ \exists = 0 \iff X \to \mathrm{Coker}(i) \to Y \to \mathrm{Coker}(f)$

$\qquad = \pi \circ f = 0.$

The assumption is: $\mathrm{Coker}(i) \to \mathrm{Ker}(\pi)$ is an isomorphism.

PS **Exercise 1** Formulate the definition of Ker(f) via universal property and prove that if exists, it is unique up to iso.

5 min ~ 10:43

Ex 2 Give an example of a non-abelian additive category.

Universal property discription.

$$\begin{tikzcd} & X \arrow[r, "f"] & Y \\ Z \arrow[ur, "i"] & W \arrow[l, "h"] \arrow[u, "g"'] & \end{tikzcd}$$

1) $f \circ i = 0$

2) $\forall g: W \to X$ s.t. $f \circ g = 0$

$\exists! h: W \to Z$ s.t. $g = i \circ h$.

$$\begin{tikzcd} & X \arrow[r, "f"] & Y \\ Z \arrow[ur, "i"] \arrow[r, shift left, "h_1"] & Z' \arrow[u, "i'"'] \arrow[l, shift left, "h_2"] & \end{tikzcd}$$

$f \circ i = 0$

$\exists!\ h_1$ s.t. $i = i' \circ h_1$

$\exists!\ h_2$ s.t. $i' = i \circ h_2$

$i = i \circ h_2 \circ h_1$

$$\begin{tikzcd} & X \arrow[r, "f"] & Y \\ Z \arrow[ur, "i"] & Z \arrow[l, "Id_Z{,}\ h_2 \circ h_1"] \arrow[u, "i"'] & \end{tikzcd}$$

Id_Z, $h_2 \circ h_1$ both make the diagram commutative.

by the uniqueness $Id_Z = h_2 \circ h_1$.

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Similarly $h_1 \circ h_2 = Id_{Z'}$ $\Rightarrow$ Z is iso to Z'.

Exercise 2 Non-abelian, additive category.

are (probably) abelian:

- Banach spaces? $X \xrightarrow{f} Y$ $\operatorname{Ker} f$ is a Banach space, $\operatorname{Coker} f = Y/\overline{\operatorname{Im} f}$ will work.
- Topological modules / top rings

take Vect_K (K a field)
remove 1-dim vector spaces $=: C$

Take $X \xrightarrow{f} Y$ $\dim X, Y > 1$
$\dim \operatorname{Ker} f = 1$ (in the usual sense).

Want to show
there is no $\operatorname{Ker}(f)$ in C.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r] \arrow[dr, "\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}"'] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R}^2 \\ & \operatorname{Ker}(f) \arrow[u] & \end{tikzcd}$$

$f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\operatorname{Ker}(f) \to \mathbb{R}^2$ injective
$\Rightarrow \dim(\operatorname{Ker} f) = 0$

$\operatorname{Ker} f = 0$

contradiction.

$$\boxed{\text{Homework: Find a category which is additive, Ker, Coker exist, but the assumption } \operatorname{Ker}(\operatorname{Coker}) = \operatorname{Coker}(\operatorname{Ker}) \text{ fails.}}$$