Remark In the definition of additive category we started by considering a category where Homs are abelian groups, and then being additive was a property of the category.

It turns out we can start with arbitrary category C, under some assumptions its Homs will automatically be equipped with the group structure. (see wikipedia, additive categories).

1) assume C has initial object (object 0 s.t. $\forall$ object X $\mathrm{Hom}(0, X)$ has 1 element)

2) ——"—— terminal object ($*$ s.t. $\forall$ object X $\mathrm{Hom}(X, *)$ has 1 element.

3) assume $0 \cong *$ (in additive categories $0 = * = 0$)

4) assume C has finite products, coproducts, products are isomorphic to coproducts.

coproducts
$\|$
direct sum

product: $X, Y \to X \times Y$ has projections $X \times Y \to X$, $X \times Y \to Y$

$\mathrm{Hom}(Z, X \times Y) = \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y)$.

(direct sum)
$\|$
coproduct: $X, Y \to X \sqcup Y$, $X \xrightarrow{i_1} X \sqcup Y \xleftarrow{i_2} Y$

$\mathrm{Hom}(X \sqcup Y, Z) = \mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z)$

: $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ what is $f+g$?

$$\boxed{X \xrightarrow{(f,g)} Y \times Y \simeq Y \sqcup Y \xrightarrow{(id, id)} Y}$$

product   coproduct

1

note in 4) we ask this map to be iso: $X \sqcup Y \to X \times Y$

$X \sqcup Y \to X$: $X \xrightarrow{id} X$, $Y \to 0 \to X$

$X \sqcup Y \to Y$: $X \to 0 \to Y$, $Y \xrightarrow{Id_Y} Y$

↳ matrix multiplication knows about addition!

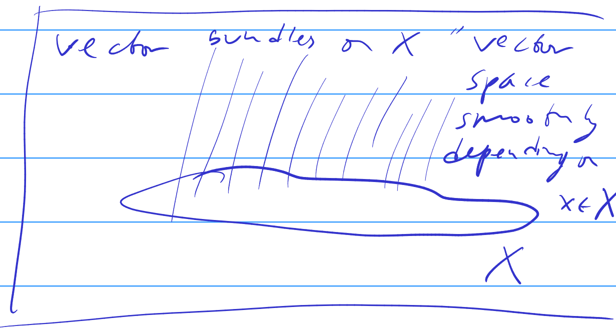
$$\boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = f+g} \quad \boxed{2}$$

Today: sheaves of modules.

Motivational Example: A commutative ring, consider the category A-mod of modules $/A$. This is an abelian category.

We will extend this definition to construct the category of quasi-coherent sheaves on any scheme X.

Construction takes steps:



Step 1: Presheaf on X with values in C

(X top. space) (C category, for instance Ab ← abelian groups)

Def Presheaf on X with values in C is a functor from $\underbrace{\{\text{open sets of } X\}}^{\text{op}} \to C$.

this is the category whose objects are open sets of X, morphisms are inclusions.

op → the functor is contravariant.

In other words, F is a presheaf if

$\forall\ U \subset X$ open we have an object $F(U)$ of C,

if $U \subset V \subset X$ are open we have restriction

$$\rho_{U,V} : F(V) \to F(U)$$

such that

if $U = V$: $\rho_{U,U} = Id$

$U \subset V \subset W$: $\rho_{U,W} = \rho_{U,V} \circ \rho_{V,W}$.

Step 2 A sheaf of abelian groups. $(\mathcal{C} = Ab)$

$\forall$ covering $\quad V = \bigcup_{\lambda \in \Lambda} U_\lambda \quad$ ($V \subset X$ open) $\qquad U_\lambda \subset X$ open

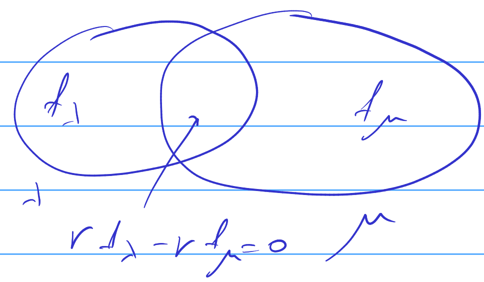
consider the map $\qquad$ abelian groups.

$$f : \mathcal{F}(V) \longrightarrow \prod_{\lambda \in \Lambda} \mathcal{F}(U_\lambda)$$

The sheaf condition: $\quad f =$ kernel of the map

$$\prod_{\lambda \in \Lambda} \mathcal{F}(U_\lambda) \to \prod_{\lambda,\mu \in \Lambda} \mathcal{F}(U_\lambda \cap U_\mu)$$

$$\{f_\lambda \mid \lambda \in \Lambda\} \to \{ r_{U_\lambda \cap U_\mu, U_\lambda} f_\lambda - r_{U_\lambda \cap U_\mu, U_\mu} f_\mu \mid \lambda, \mu \in \Lambda \}.$$



Question do sheaves form an abelian category?

0 sheaf : $\quad \mathcal{F}(U) = \{0\}$

direct sums : $\quad \mathcal{F}, \mathcal{G} \in \underline{Sh_{Ab}(X)} \qquad (\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$.

sheaves of abelian groups on X

$U \subset V$

the sheaf axiom is checked componentwise $\qquad \rho^{\mathcal{F}\oplus\mathcal{G}}_{U,V} = \rho^{\mathcal{F}}_{U,V} \oplus \rho^{\mathcal{G}}_{U,V}$

(elements of $(\mathcal{F} \oplus \mathcal{G})(U)$ are pairs (f, g)).

$$\rho^{\mathcal{F}\oplus\mathcal{G}}_{U,V} \left\{ \begin{array}{ccc} \mathcal{F}(V) & \oplus & \mathcal{G}(V) \\ \downarrow \rho^{\mathcal{F}}_{U,V} & & \downarrow \rho^{\mathcal{G}}_{U,V} \\ \mathcal{F}(U) & \oplus & \mathcal{G}(U) \end{array} \right.$$

Homs: $\mathrm{Hom}(F, G) = \{ F(U) \to G(U)$ $\forall$ open $U \subset X \}$ such that for $V \subset U$:

$$\begin{tikzcd} F(U) \arrow[r, "\varphi_U"] \arrow[d, "\rho^F_{V,U}"'] & G(U) \arrow[d, "\rho^G_{V,U}"] \\ F(V) \arrow[r, "\varphi_V"] & G(V) \end{tikzcd}$$
is commutative

Notation: $\varphi \in \mathrm{Hom}(F, G)$ corresponds to $\varphi_U : F(U) \to G(U)$

Kernels

Prop If $\varphi : F \to G$ is a map of sheaves.

Let $K(U) = \mathrm{Ker}(\underbrace{F(U) \to G(U)}_{\varphi_U})$.

Then K is a sheaf

Proof

$V \subset U$

$$\begin{tikzcd} K(U) = \mathrm{Ker}(F(U) \arrow[d] \arrow[r] & G(U)) \arrow[d] \\ K(V) = \mathrm{Ker}(F(V) \arrow[r] & G(V)) \end{tikzcd}$$

If $U = \bigcup_\lambda V_\lambda$

$$\begin{tikzcd} K(U)^{(1)} \arrow[r] \arrow[d, phantom, "\cap"] & \prod K(V_\lambda)^{(2)} \arrow[d, phantom, "\cap"] \\ F(U)^{(3)} \arrow[r] & {}^{(4)}\prod F(V_\lambda) \end{tikzcd}$$

injective by the sheaf condition for F.

0 in (2) $\Rightarrow$ 0 in (4) $\Rightarrow$ 0 in (3) $\Rightarrow$ 0 in (1)

so $K(U) \to \prod K(V_\lambda)$ is injective.

Suppose we have $f_\lambda \in K(V_\lambda)$ such that

$$\begin{tikzcd} K(U)^{(6)} \arrow[r] \arrow[d, phantom, "\cap"] & \prod_\lambda K(V_\lambda)^{(1)} \arrow[r] \arrow[d, phantom, "\cap"] & \prod_{\lambda,\mu} K(V_\lambda \cap V_\mu)^{(2)} \arrow[d, phantom, "\cap"] \\ F(V)^{(5)} \arrow[r] \arrow[d] & \prod_\lambda F(V_\lambda)^{(4)} \arrow[r] \arrow[d] & {}^{(3)}\prod F(V_\lambda \cap V_\mu) \\ G(V)^{(7)} \arrow[r] & \prod_\lambda G(V_\lambda)^{(8)} & \end{tikzcd}$$

some $x \in (1)$ goes to 0 in (2) $\Rightarrow$ go to 0 in (3) $\Rightarrow$ (4) comes from (5).

is (5) coming from (6)?

This is true if (5) goes to 0 in (7).

(7) is 0 $\Leftrightarrow$ (8) is 0. (by sheaf axiom for G)

(8) comes from (4) from (1)

since $K(V_\lambda) = \mathrm{Ker}(F(V_\lambda) \to G(V_\lambda))$ we see that (8) = 0.

∎

What about cokernels? $\quad \operatorname{coker}(X \xrightarrow{f} Y) = Y/\operatorname{Im} f$.

<u>Problem</u> for $\varphi : F \to G$

Let $C(U) = \operatorname{coker}(F(U) \to G(U))$ then $C(U)$ doesn't have to be a sheaf.

$U = \bigcup_\lambda V_\lambda$

$$\begin{tikzcd}
F(U) \arrow[r, "(6)"'] \arrow[d] & \prod_\lambda F(V_\lambda)^{(5)} \arrow[r] \arrow[d] & \prod_{\lambda,\mu} F(V_\lambda \cap V_\mu) \arrow[d, "(7)"] \\
G(U)_{(3)} \arrow[r] \arrow[d] & \prod_\lambda G(V_\lambda)^{(4)} \arrow[r] \arrow[d] & \prod_{\lambda,\mu} G(V_\lambda \cap V_\mu)^{(8)} \arrow[d] \\
C(U)^{(1)} \arrow[r] & \overset{(2)}{\prod_\lambda} C(V_\lambda) \arrow[r] & \prod_{\lambda,\mu} C(V_\lambda \cap V_\mu)
\end{tikzcd}$$

is it injective?

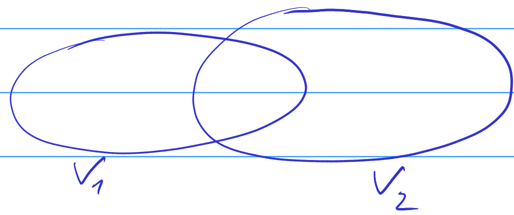
(1) goes to 0 in (2)

(1) comes from (3)

$\Rightarrow$ (4) goes to 0 in (2) $\Rightarrow$ comes from (5)

want: (3) comes from (6) $\qquad$ we need (6)!

we need (5) goes to 0 in (7)
we only know that 5 goes to 0 in (8).



$f \in C(U)$ is zero on V_1 if it comes from $F(V_1)$, on V_2 if it comes from $F(V_2)$. it is not true in general that it means f comes from $F(V_1 \cup V_2)$, we don't know if this el-s of $F(V_1)$, $F(V_2)$ glue nicely on the intersection of $V_1 \cap V_2$.

The main construction to remedy this problem is <u>sheafification</u>. ($C(U)$ is a presheaf, we will sheafify it).

Construction

Input:
$\mathcal{F}$ presheaf ($\mathcal{F}(U)$, restriction maps, no sheaf axioms)

(Idea: $\mathcal{F}$ should be "corrected" on small open sets)

We will construct a sheaf $\mathcal{F}^{sh}$
$\forall U$ $\mathcal{F}^{sh}(U)$ is the quotient $\frac{A}{B}$
where

$$A = \bigoplus_{\substack{\text{all coverings of} \\ U \text{ by open sets} \\ U = \bigcup V_\lambda}} \operatorname{Ker}\left(\prod_\lambda \mathcal{F}(V_\lambda) \to \prod_{\lambda,\mu} \mathcal{F}(V_\lambda \cap V_\mu) \right)$$

$$B = \quad (f_\lambda)_{\lambda \in \Lambda} - (r_{V_{\lambda,\mu}, V_\lambda} f_\lambda)_{\lambda,\mu \in \Lambda}$$

$U = \bigcup_\lambda V_\lambda$

$V_\lambda = \bigcup_\mu V_{\lambda,\mu}$

$f_\lambda \in \mathcal{F}(V_\lambda)$

$$\mathcal{F} \to \mathcal{F}^{sh} \xrightarrow{\exists !} \mathcal{G} \text{ sheaf}$$

$$\operatorname{Hom}(\mathcal{F}^{sh}, \mathcal{G}) = \operatorname{Hom}(\mathcal{F}, \mathcal{G})$$