Last time: $f: X \to Y$ $\quad f^{-1}$, f_*

Later we see $\mathrm{Hom}(f^{-1}F, G) \cong \mathrm{Hom}(F, f_* G)$.

Today: <u>exactness</u>

Important language!

Let $\mathcal{A}$ be an abelian category, for instance modules/some ring.

<u>Def</u> A <u>chain</u> complex is a sequence of objects C_i ($i \in \mathbb{Z}$), differentials $d_i : C_i \to C_{i-1}$ satisfying $d^2 = 0$:

$$0 = d_{i-1} \circ d_i : C_i \to C_{i-1} \to C_{i-2}$$

<u>Cochain</u> complex is similar $\{C^i\}_{i \in \mathbb{Z}}$ differentials $d^i : C^i \to C^{i+1}$ satisfying $d^{i+1} d^i = 0$.

$C_i = C^{-i}$ equivalent

<u>Def</u> Given a chain complex (C_i, d_i) the homology is

$$H_i(C_\cdot, d_\cdot) = \frac{\mathrm{Ker}(d_i : C_i \to C_{i-1})}{\mathrm{Im}(d_{i+1} : C_{i+1} \to C_i)}$$

in the abelian category language:

$$\begin{tikzcd}
C_{i+1} \arrow[r, "d_{i+1}"] \arrow[dr, "(2)"'] & C_i \arrow[r, "d_i"] & C_{i-1} \\
& K = \mathrm{Ker}(d_i) \arrow[u, "(1)"'] \arrow[r] & H_i
\end{tikzcd}$$

$H_i = \mathrm{Coker}\,(2)$

(1) is comes with kernel

(2) is by the univ. property of Ker

<u>Def</u> a complex is called exact in term i if $H_i = 0$. Equivalently $\mathrm{Im}\, d_{i+1} = \mathrm{Ker}\, d_i$ using abelian category language:

$$\begin{tikzcd}
C_{i+1} \arrow[r] \arrow[dr, "(2)"'] & C_i \arrow[r] & C_{i-1} \\
& K \arrow[u, "(1)"'] &
\end{tikzcd}$$

Examples

$0 \to X \xrightarrow{f} Y$

exact means

$\mathrm{Ker}\, f = 0$, i.e. f is injective

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[r] & 0 \\ & K = Y \arrow[u] & \end{tikzcd}$$

exact here means f is surjective.

a little longer

$0 \to X \to Y \to Z$

exact here and here means: $X = \mathrm{Ker}\,(Y \to Z)$

left exact

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[rr] \arrow[dr, "(\cong)"'] & & Y \arrow[r] & Z \\ & & K \arrow[ur, hook] & & \end{tikzcd}$$

right exact

$X \to Y \to Z \to 0$

exact means $Z = \mathrm{Coker}\,(X \to Y)$.

short exact sequence

last example: $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ exact everywhere means $f = \mathrm{Ker}\,(g)$, $g = \mathrm{Coker}\, f$, means $Z = Y/X$.

Exact functors

Suppose we have a covariant functor $\varphi : \mathcal{A} \to \mathcal{A}'$ between abelian categories, we say φ is left/right exact if it sends left exact sequences to left exact, respectively right $\to$ right.

If φ is contravariant then φ is left exact if it sends right exact seq. to left exact.

(in the context of additive categories all functors are assumed linear)

<u>Equivalently</u> φ covariant is left exact if it sends kernels to kernels, right exact if it sends cokernels to cokernels

φ contravariant is left exact if it sends cokernels to kernels.

<u>Example</u>

Hom is left exact (in both ways).

What does it mean?

Suppose $X \in \mathcal{A}$ is fixed. Then for any $Y \in \mathcal{A}$

$Hom(X,Y)$ is an abelian group (or a vector space if we are working with k-linear categories for some field k)

$Y \to Hom(X,Y)$ is a covariant functor from $\mathcal{A}$ to Ab.

Suppose

$0 \to K \to Y \xrightarrow{f} Z$ is left exact,

i.e. $K = Ker(f)$. Recall the definition:

$Hom(X,K) \subset Hom(X,Y)$ consisting of maps $X \to Y$ which become zero when composed with f.

$\Leftrightarrow$ $Hom(X,K) = Ker(Hom(X,Y) \to Hom(X,Z))$.

So $Hom(X,-)$ sends kernels to kernels, so is left exact.

Similarly consider functor $Y \to Hom(Y,X)$

composition works: $Y \to Z$ $\quad Hom(Z,X) \to Hom(Y,X)$.

So it is a contravariant functor.

for $Y \xrightarrow{f} Z \to K$ $\quad Coker(f) = K$ is such that

$Hom(K,X) \subset Hom(Z,X)$ which become zero..

so $Hom(K,X) = Ker(Hom(Z,X) \to Hom(Y,X))$.

So $Hom(-,X)$ sends cokernels to kernels.

So is also left exact.

<u>Rule of thumb</u>: functors defined by some universal properties are usually left or right exact, but not both.

Going back to sheaves. $f : X \to Y$

$F \in Sh(X)$ $\quad$ $G \in Sh(Y)$

means sheaves on ...

let's look at

$Hom(f^{-1}G, F)$

Recall $\quad f^{-1}G = \underbrace{\left(\begin{matrix} U \\ \cap \\ X \end{matrix} \to \lim_{V \supset f(U)} G(V) \right)}_{\text{presheaf}}^{sh}$ $\qquad$ $F \quad G$, $\quad f : \begin{matrix} X \\ \cup \\ U \end{matrix} \to Y$

$$Hom((-)^{sh}, F) = Hom(-, F) = \left\{ \lim_{V \supset f(U)} G(V) \to F(U) \right\}_{\forall\, U \subset X}$$

$$\| $$

$$\forall V, \forall U, V \supset f(U) \quad \{ G(V) \to F(U) \}$$

(compat. with restrictions)

$V \supset f(U) \iff f^{-1}(V) \supset U$

$$\|$$

now fix V, go over all U. $\qquad \{ G(V) \to F(f^{-1}(V)) \}$

any other U we have

$$G(V) \to F(f^{-1}(V)) \xrightarrow[\text{restriction.}]{} F(U)$$

__Corollary__ a morphism $f^{-1}(G) \to F$ is the same thing as a collection of morphisms (compat. with restrictions)

$G(V) \to F(f^{-1}(V))$ for all open $V \subset Y$.

So it is just $G \to f_* F$.

Exercise: show that f_* is left exact, f^{-1} is right exact.

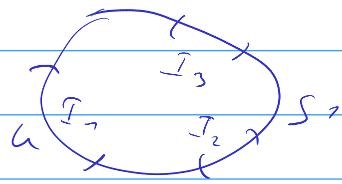
$$C^\infty(S^1,\mathbb{R}) = \{ f\in C^\infty(\mathbb{R},\mathbb{R}) \mid f(x+1)=f(x) \ \forall\, x\in\mathbb{R}. \}$$

$C^\infty(S^1,\mathbb{R}) \xrightarrow{\frac{d}{dx}} C^\infty(S^1,\mathbb{R})$. Denote the sheaf by F.

$\mathrm{Ker}\left(\frac{d}{dx}\right)$ = locally constant functions.

$\forall\, U\subset S^1 \quad \mathrm{Ker}\left(\frac{d}{dx}\right)(U) = \{\text{locally constant f-s on } U\}$

$\mathrm{Ker}\left(\frac{d}{dx}\right)(U) = \mathrm{Ker}\left(F(U)\xrightarrow{\frac{d}{dx}} F(U)\right)$



$U = I_1\cup I_2\cup I_3$

a function f is locally constant $\iff f'=0$.

can be different constants on I_1, I_2.

$\mathrm{Ker} = \underline{\mathbb{C}}$: $\underline{\mathbb{C}}(U) = \prod_{\substack{\text{connected}\\ \text{components of } U}} \mathbb{C}$.

Question 2 what is the cokernel?

if $U\subset S^1$ not everything

if U is connected $f(U)$ has integral $\int f$ (indefinite)

$U\cong(a,b)$

$d\int f = f$.

otherwise maybe $\neq 0$. so $\left(\mathrm{coker}\frac{d}{dx}\right)(U)=0$

Claim $\mathrm{coker}\frac{d}{dx} = 0$. Take $U\subset S^1$ open

$\exists$ a cover of U by open intervals (a_λ, b_λ) as above.

Pick $s\in \mathrm{coker}\frac{d}{dx}(U)$ $\quad s|_{(a_\lambda,b_\lambda)} = 0 \Rightarrow s=0$.

new notation for restriction

It turns out $\frac{d}{dx}: \mathcal{F} \to \mathcal{F}$ is surjective!

Exercise

Exercise : 1) show that f_* is left exact, f^{-1} is right exact.

using

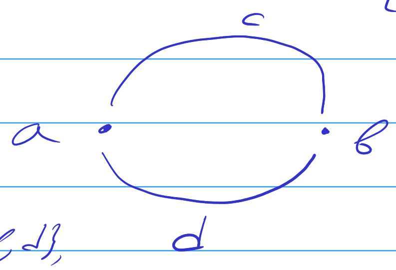
$$\mathrm{Hom}(f^{-1}G, F) = \mathrm{Hom}(G, f_* F)$$

2) Examples of f_* f^{-1}

a) what is f_*, f^{-1} for a1) $f: X \to \text{point}$

a2) $f: \text{point} \to X$.

b) More explicitly take $X = \{a, b, c, d\}$



topology:

a, b closed

$\overline{\{c\}} = \{c, a, b\}$

$\overline{\{d\}} = \{d, a, b\}$

So closed sets are :

$\emptyset$, $\{a\}$, $\{b\}$, $\{a,b,c\}$, $\{a,b,d\}$, $\{a,b\}$, $\{a,b,c,d\}$

(7 closed sets, 7 open sets)

open sets : $\emptyset$, $\{a,b,c,d\}$, $\{c\}$, $\{d\}$, $\{c,d\}$, $\{c,b,d\}$, $\{c,a,d\}$.

$\boxed{X \text{ is like circle}}$ Question: What is a sheaf on X?, What do these functors in a) do?

is f_* right exact ?

is f^{-1} left exact ?