Key idea: if some functor is not exact, we should measure its non-exactness, and hope that the result is an in interesting invariant. So how do we measure non-exactness? We construct derived functors!

Motivating example: $X = S^1$. We have seen that there is a s.e.s. $\quad t \quad$ dt S_n

$$0 \to \mathbb{R}_{S^1} \longrightarrow C^\infty(S^1) \xrightarrow{\frac{d}{dt}} C^\infty(S^1) \to 0$$

locally constant functions

apply π_* $\quad$ $\pi : S^1 \to \text{point}$

generally $\quad \pi_*(F) = F(X)$. So applying π_* also denotes $\Gamma(X, F)$ $\quad (\Gamma)$

we get $\quad 0 \to \mathbb{R} \to C^\infty(S^1, \mathbb{R}) \xrightarrow{\frac{d}{dt}} C^\infty(S^1, \mathbb{R})$

not surjective! in fact the cokernel is

$$C^\infty(S^1; \mathbb{R}) \Big/ \underbrace{\tfrac{d}{dt} C^\infty(S^1, \mathbb{R})}_{\text{functions whose integral is zero.}} \xrightarrow[\text{the map is the integral!}]{\simeq} \mathbb{R}$$

The failure to exactness is measured by

$$\mathbb{R} = \int f = H^1(S^1, \mathbb{R}).$$

More generally, suppose we have a functor which is left exact, but not exact, for instance f_* for $f : X \to Y$.

If we have $0 \to F \to G \to H \to 0$ s.e.s. of sheaves on X,

apply π_*: take the cokernel of the last map

$$0 \to \pi_*(F) \to \pi_*(G) \to \pi_*(H) \longrightarrow C_{F,G,H} \to 0$$

Let's try to understand C_{FGH}.

<u>Def</u> a SES as above is <u>split</u> if G is isomorphic to $F \oplus H$ in such a way that $F \to G$ is the inclusion $F \to F \oplus H$ and $G \to H$ is the projection $F \oplus H \to H$.

<u>Prop</u> this is equivalent to either of the following:

1) $\exists\ H \to G$ s.t. $H \to G \to H = Id_H$
2) $\exists\ G \to F$ s.t. $F \to G \to F = Id_F$

this new map is called section.

$$0 \to F \to G \to H \to 0$$

<u>Proof</u> easy. in 1)

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "f"] & G \arrow[r, "g"'] & H \arrow[l, bend right, "s"'] \arrow[r] & 0 \end{tikzcd}$$

Define $s': G \to F$ by $\underbrace{Id_G - s\circ g}_{\text{image in } F}$

$$G \underset{\sim}{\xrightarrow{(s', g)}} F \oplus H \underset{\sim}{\xrightarrow{(f, s)}} G.$$

2) is similar.

<u>Observation</u> If SES is split, then $C_{F,G,H} = 0$.

because $G = F \oplus H$, so $\pi_*(G) = \pi_*(F) \oplus \pi_*(H)$, hence $0 \to \pi_*(F) \to \pi_*(G) \to \pi_*(H) \to 0$ is exact.

<u>Observation</u> Suppose F is <u>injective</u>.

$Hom(-, F)$ is exact, i.e. for any injection $A \to B$, $Hom(B, F) \to Hom(A, F)$ is surjective.

i.e. any map $A \to F$ can be extended to B.

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[dl, dashed, "s"] & H \\ & F & & \end{tikzcd}$$

By injectivity Id_F can be extended to $s: G \to F$ s.t. $s|_F = Id_F$, so s is a section so SES is split, so $C_{FGH} = 0$. Hint maybe C_{FGH} depends only on F (in some sense).

This motivates the following:

Fix F, try to construct the universal C_{FGH} (if F is injective it will be zero).

Fix F, consider different G's. How does C_{FG} vary?

$$C_{FGH} = C_{F \subseteq G}$$

Suppose $F \subset I$, where I is injective. but G is arbitrary

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d, "(1)\ \exists"] & G/F \arrow[r] \arrow[d, "(2)"] \arrow[dl, dashed] & 0 \\ 0 \arrow[r] & F \arrow[r] & I \arrow[r] & I/F \arrow[r] & 0 \end{tikzcd}$$

by injecting of I $\exists$ (1) (not unique) so induces (2).

applying π_* we obtain a map $C_{F \subset G} \to C_{F \subset I}$.

$$\begin{tikzcd} 0 \arrow[r] & \pi_*(F) \arrow[r] \arrow[d] & \pi_*(G) \arrow[r] \arrow[d] & \pi_*(G/F) \arrow[r] \arrow[d] \arrow[dl] & C_{F \subset G} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \pi_*(F) \arrow[r] & \pi_*(I) \arrow[r] & \pi_*(I/F) \arrow[r] & C_{F \subset I} \arrow[r] & 0 \end{tikzcd}$$

corrections

any 2 extensions $G \to I$ differ by a map $G/F \to \cancel{F}$. $G/F \to I$

~~so (2) does not depend on the choice of extension~~, hence $C_{F \subset G} \to C_{F \subset I}$ doesn't.

So in a sense $C_{F \subset I}$ is universal! This will be our derived functor.

For instance if $F \subset I'$ another embedding with I' injective object we obtain maps $C_{F\subset I} \to C_{F\subset I'} \to C_{F\subset I}$

$$\underbrace{C_{F\subset I} \to C_{F\subset I'} \to C_{F\subset I}}_{\text{must be } Id}$$

must be Id, so $C_{F\subset I} \simeq C_{F\subset I'}$.

Continue this process as follows

$F \to I_0$, embed $I_0/_F = F^1$ into another injective I_1, $I_1/_{F_1} \subset I_2$, and so on, obtain exact sequence

$$0 \to F \to \underbrace{I_0 \to I_1 \to I_2 \to \dots}$$

this is called injective resolution of F.

apply π_* to the resolution

$$\pi_*(I_0) \to \pi_*(I_1) \to \dots$$

Def Cohomology sheaves of this complex are called the right derived functors of π_*

$$(R^k\pi_*)(F) := H^k(\pi_*(I_\cdot)).$$

In a special case when $\pi: X \to$ point, i.e. $\pi_* = \Gamma(X, -)$, we call it the cohomology of F

$$H^k(X, F) := (R^k\Gamma)(F) = H^k(I_0(X) \to I_1(X) \to \dots).$$

H.W. 1) Using $Hom(f^{-1}G, F) = Hom(G, f_* F)$.

Markers: $0 \to F \to G \to H$

$\forall \mathcal{H} \quad 0 \to Hom(\mathcal{H}, \pi_* F) \to Hom(\mathcal{H}, \pi_* G) \to Hom(\mathcal{H}, \pi_* H)$

by definition $Ker(\pi_* G \to \pi_* H)$ is such object M that $\forall \mathcal{H}$ $Hom(\mathcal{H}, M) = \{ f: \mathcal{H} \to \pi_* G \mid \mathcal{H} \to \pi_* G \to \pi_* H = 0 \}$.

so $\pi_* F = Ker(\pi_* G \to \pi_* H)$ by definition of kernel.

$f: A \to B$ $\quad Ker(f) \to A$ is such that

$\forall \mathcal{X}$ $\quad Hom(\mathcal{X}, Ker(f)) = \{ s \in Hom(\mathcal{X}, A) \mid f \circ s = 0 \}$

so <u>Lemma</u>

$0 \to A \to B \to C$ is exact $\iff$

$\forall \mathcal{X}$ $\quad 0 \to Hom(\mathcal{X}, A) \to Hom(\mathcal{X}, B) \to Hom(\mathcal{X}, C)$ is exact.

(just Definition of Ker)

a) $\quad i \quad * \to X$ $\quad i^{-1}(F) =$ stalk of F at $i(*)$

$i(*) = p$

$F \in Sh_*$ (just abelian group G)

$$i_* F(U) = \begin{cases} G & p \in U \\ 0 & p \notin U \end{cases}$$

$\pi: X \to *$ $\qquad \pi_* F = F(X)$

$F \in Sh_X$

G sheaf on $*$ $\qquad (\pi^{-1} G)(U) = \left(\lim\limits_{\substack{V \\ \pi(U) \subset V}} G(V) \right)^{sh}$

$V = \emptyset$ or point

$U \neq \emptyset \Rightarrow G(V) = G$

$\lim = G$

$U = \emptyset$

Simon:

So we have to sheafify the constant presheaf $F(U) = G$.

<u>obtain</u> locally constant maps $U \to G$.

<u>Part b)</u> $\quad a, b, c, d, . \quad a, b$ closed

$\overline{\{c\}} = \{c, a, b\}$

$\overline{\{d\}} = \{d, a, b\}$

$\emptyset$, c, d, acd, bcd, X open

0

G

$$\begin{tikzcd}
 & G(acd) \arrow[dr] & & G(c) \arrow[dr] & \\
G(X) \arrow[ur] \arrow[dr] & & G(cd) \arrow[ur] \arrow[dr] & & 0 \\
 & G(bcd) \arrow[ur] & & G(d) \arrow[ur] &
\end{tikzcd}$$

choose

$G(c)$, $G(d)$ $\qquad G(c,d) = G(c) \times G(d)$

$G(acd)$, $G(bcd)$ along with maps to $G(c,d)$

$G(X)$ determined

stalks: at c : $G(c)$ $\qquad$ at a : $G(acd)$

at d : $G(d)$ $\qquad$ at b: $G(bcd)$

X is connected $\Rightarrow$ loc. constant $\Leftrightarrow$ constant.

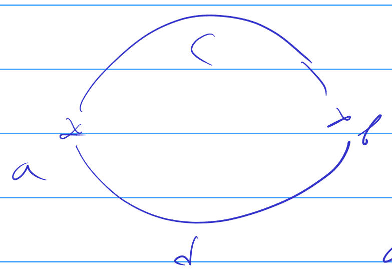
Map $G(acd)$ to $G(c,d)$ is the same as pair of maps $G(acd) \to G(c)$, $G(acd) \to G(d)$

$$\begin{tikzcd}
G(acd) \arrow[r] \arrow[dr] & G(c) \\
G(bcd) \arrow[r] \arrow[ur] & G(d)
\end{tikzcd}
\qquad
\begin{tikzcd}
G(acd) \arrow[r] \arrow[d] & G(c) \\
G(d) & G(bcd) \arrow[l] \arrow[u]
\end{tikzcd}$$

sheaves are diagrams.

$(i_a)_* G$ $\qquad \begin{matrix} G & 0 \\ 0 & 0 \end{matrix}$

$(i_c)_* G$ $\qquad \begin{tikzcd} G \arrow[r] & G \\ 0 & G \arrow[u] \end{tikzcd}$.



functions on S^1 are pairs f on I

$acd \sim S^1 \setminus b$

$bcd \sim S^1 \setminus a$

function on $S^1 \setminus b$ and on $S^1 \setminus a$, which coincide on c,d are functions on S^1.

<u>Exercises</u> What does it mean for an abelian group to be an injective object? (in Ab).