FUKAYA CATEGORY

M^{2n} smooth manifold $\qquad (M,\omega)$ symplectic manifold

$\omega \in \Omega^2(M)$ symplectic form (non-degenerate & closed)

$\omega^n \neq 0$

$d\omega = 0$

examples: 1) $\mathbb{R}^{2n}$ $(x_1, y_1, \dots, x_n, y_n)$ $\qquad \omega = \sum_{i=1}^{n} dx_i \wedge dy_i$

2) cotangent bundle T^*Q

3) Kähler manifolds (X, g)

$\omega(X,Y) = g(iX, Y)$

$d\omega = 0$

quasi projective varieties

$S \subset M$ submanifold

→ isotropic if $\omega|_S = 0$ $\quad (\Rightarrow \dim S \leq n)$

→ Lagrangian if isotropic + $\dim = n$

Hamiltonian vectorfield:

$H \in C^\infty(M, \mathbb{R})$ $\quad$ $H : M \to \mathbb{R}$ $\quad \rightsquigarrow dH \in \Omega^1(M)$

$X_H \in \mathfrak{X}(M)$ defined by $\omega(\cdot, X_H) = dH(\cdot)$

vectorfield

$H \in C^\infty(M \times \mathbb{R}_t, \mathbb{R})$ ← time - dependent

$\rightsquigarrow \quad H_t = H(\cdot, t) \quad \rightsquigarrow \quad X_{H_t}$

φ_t is the flow of X_{H_t}

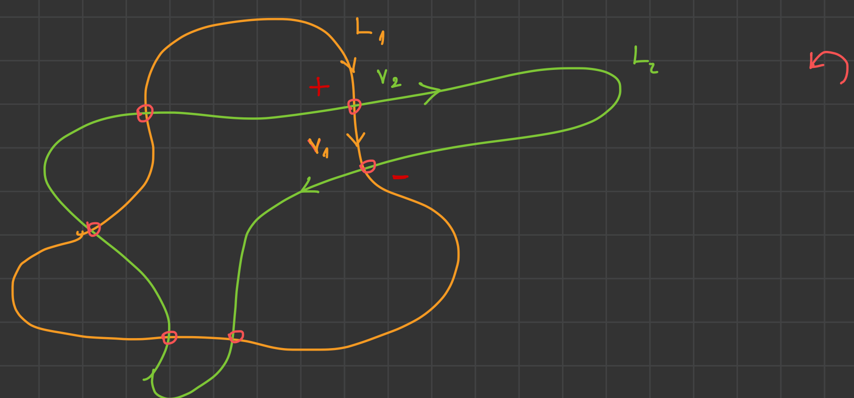
$\varphi_t : M \longrightarrow M$ is a <u>Hamiltonian isotopy</u>

<u>Prop</u>: φ_t is a Ham isotopy $\quad \varphi_t^* \omega = \omega$

<u>Thm</u>: $\mathrm{Ham}(M,\omega) \lneq \mathrm{Symp}(M,\omega) \subseteq \mathrm{Diff}(M)$

$L_1^n, L_2^n \subset M^{2n}$ $\quad$ make $L_1 \pitchfork L_2$

oriented $\quad$ oriented $\quad$ transverse



intersection # : $I(L_1, L_2)$ = signed # of intersections

Lagrangian Floer homology: intersection theory for Lagrangian submanifolds.

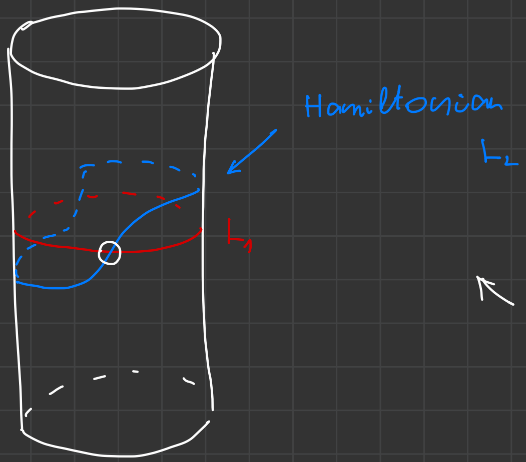
$I(L_1, L_2) \in \mathbb{Z}$ $\rightsquigarrow$ vectorspace formally $\underbrace{\text{gen by intersection}}$

$\|$ recover

Euler characteristic

$\oplus C_i$

grading (for now $\underline{\underline{\mathbb{Z}_2}}$)



could be made disjoint by $\underline{\text{smooth}}$ isotopy

$\Rightarrow I(L_1, L_2) = 0$

$T^*S^1 \cong S^1 \times \mathbb{R}$

$HF^*(L_1, L_2)$ has rk 2 $\rightarrow$ odd rk 1, $\rightarrow$ even rk 1

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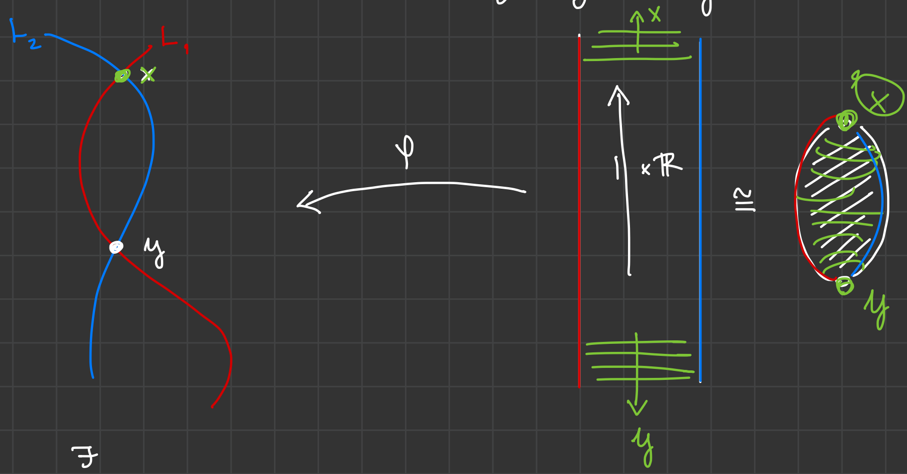
idea of def of $HF^*(L_1, L_2)$

$CF^*(L_1, L_2)$ = formally gen by $L_1 \cap L_2$



count the # of mappings from $\mathbb{R} \times I \longrightarrow M$ $/\mathbb{R}$

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inspiration for def is from Morse homology

$$\Omega(L_1, L_2) = \{ \gamma : I \to M \mid \gamma(0) \in L_1, \gamma(1) \in L_2 \}$$

$$\mathcal{A} : \Omega(L_1, L_2) \longrightarrow \mathbb{R}$$

$$d\mathcal{A}_\gamma(X) = \int_0^1 \omega(X, \dot{\gamma})\, dt$$

Use of HF^*

→ displacebility of Lagrangians

→ fixed pts of symplectic diffeom

→ understand symplectomorphisms throug Lagrangians

→ Seiberg-Witten homology

Heegaard-Floer hom

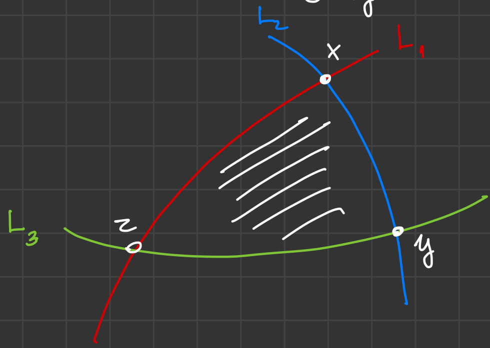
Instanton-homology

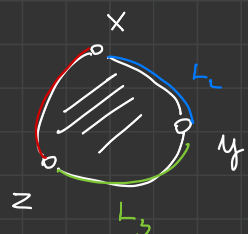
Khovanov-homology

- symplectic Picard-Lefshetz Thm

$$HF^*(L_1, L_2) \otimes HF^*(L_2, L_3) \longrightarrow HF^*(L_1, L_3)$$

$$x \otimes y$$

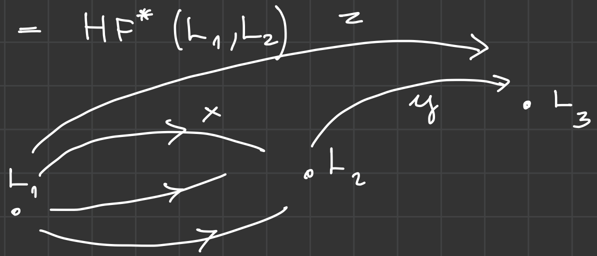




Category (Donaldson)

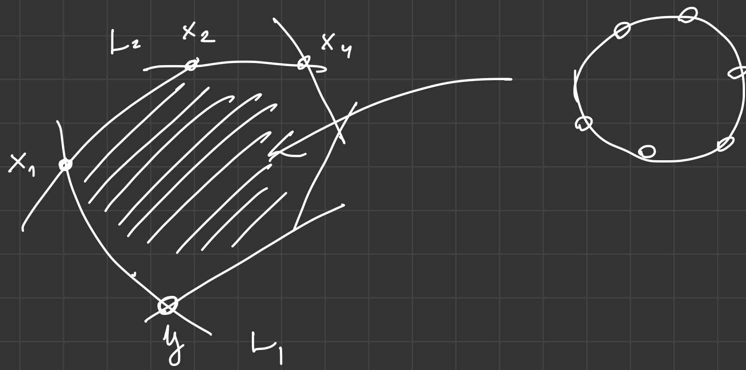
Object = Lag

Mor = $HF^*(L_1, L_2)$



Fukaya category

$$HF^*(L_1, L_2) \otimes HF^*(L_2, L_3) \otimes \cdots \otimes HF^*(L_{k-1}, L_k) \to HF^*(L_1, L_k)$$



A_∞ -structure

Mirror symmetry:

1, Morse theory (Morse hom) 2 lect

2, Symplectic mfds 3 lect

3, Lagrangian Floer hom

Overview

def

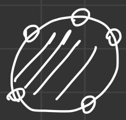
2-dim

(Dehn twist)

4, A_∞ - categories

- Deligne - Mumford compactification

of the bdry punctured disc



- Stasheff associahedra

- A_∞ - algebras & categories

5, Fukaya category

6, research talk on our level in the

subject. (James Pascaleff)

$$
\begin{tikzcd}
1 \arrow[r] & 3 \arrow[dr] & & \\
2 \arrow[ur] \arrow[rr] & & 5 \arrow[r] & 6 \\
4 \arrow[urr] & & &
\end{tikzcd}
$$

Write to Anton about topic choice

by end of this week March 7 23^{59}