We were describing all holomorphic sections of $O(m)$ on $\mathbb{C}P^n$

(truly speaking ~~holomorphic~~ regular = given by polynomial (rational functions))

each section on U_0 is given by a polynomial $P_0(x_1, \dots, x_m)$ of total degree $\leq m$

Homogeneous description:

$$P_0 \Rightarrow P(x_0, \dots, x_n) = x_0^m P_0(x_1/x_0, \dots, x_n/x_0).$$

P is homogeneous of degree $= m$ in $n+1$ variables.

given P:

on U_i: $P_i = P(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$

Hirzebruch - Riemann - Roch:

Connect chern classes and dim of space of cross-sections.

For nice vector bundles it says:

$$\int_{[X]} Td_X \, ch(\mathcal{E}) = \chi R\Gamma(X, \mathcal{E}).$$

X: compact complex manifold.

$\mathcal{E}$: holomorphic vector bundle

Right hand side:

$$\chi R\Gamma(X, \mathcal{E}) = \sum_{i=0}^{\dim X} \dim H^i(X, \mathcal{E})(-1)^i$$

$H^i(X, \mathcal{E}) =$ "i-th cohomology of X with coefficients in $\mathcal{E}$".

$H^0(X, \mathcal{E}) \overset{\text{def}}{=} \Gamma(X, \mathcal{E}) \overset{\text{def}}{=}$ vector space of all holomorphic cross sections (for algebraic varieties like $\mathbb{C}P^n$ we can replace by regular cross sections).

for "nice" $\mathcal{E}$, with "many" cross-sections $H^i(X, \mathcal{E}) = 0$ $(i > 0)$.

Use $H^i(X, \mathcal{E})$ as a black box for now.

for $m \geq 0$

$$\chi R\Gamma(\mathbb{C}P^n, O(m)) = \dim H^0(X, \mathcal{E})$$

$$\dim \begin{pmatrix} \text{homogeneous polynomials} \\ \text{in } n+1 \text{ variables} \\ \text{of degree } m \end{pmatrix} = \dim \Gamma(X, \mathcal{E})$$

$$\| \qquad\qquad \binom{n+m}{m}$$

$\mathbb{P}^1$ $n = 1$ get $m+1$

$m = 1 \rightarrow n+1$

This is the RHS of HRR.

Let us define LHS.

$\int \mathrm{Td}_X \, ch(\mathcal{E})$

$[X]$ p

Out of $\{c_i(\mathcal{E})\}$ we can construct other invariants as follows.

Suppose $\mathcal{E} = L_1 \oplus \ldots \oplus L_r$, where L_i is a line bundle.

Then

$$c(\mathcal{E}) = \sum_{i=0}^{\dim X} c_i(\mathcal{E}) = \prod_{j=1}^{r} \left(1 + c_1(L_j)\right)$$

(from $c_k(\mathcal{E} \oplus \mathcal{E}') = \sum_{i=0}^{k} c_i(\mathcal{E}) \cup c_{k-i}(\mathcal{E}')$

$$\Downarrow$$

$$c(\mathcal{E} \oplus \mathcal{E}') = c(\mathcal{E}) \cup c(\mathcal{E}') \text{).}$$

alternatively, $c_i(\mathcal{E}) = e_i\left(c_1(L_1), c_1(L_2), \ldots\right)$ (*)

where $e_i(x_1, \ldots, x_r) = \sum_{j_1 < \ldots < j_i} x_{j_1} \ldots x_{j_i}$

the elementary symmetric function

$$e_1(x_1, \ldots, x_r) = x_1 + \ldots + x_r$$

$$e_2(x_1, \ldots, x_r) = x_1 x_2 + x_1 x_3 + \ldots + x_1 x_r + x_2 x_3 + \ldots x_2 x_r + \ldots x_{r-1} x_r.$$

basic algebra:

$(1+x_1)(1+x_2)\ldots(1+x_r) =$ sum of all product of subsets of $x_1, \ldots, x_r = 1 + e_1 + e_2 + \ldots + e_r$.

Therefore the formula.

The idea is to consider other kinds of functions instead of $e_1, e_2, \ldots$.

Define $ch(\mathcal{E})$, called Chern character by

$$ch(\mathcal{E}) = \underbrace{e^{c_1(L_1)}}_{\text{has no meaning,}} + \ldots + e^{c_1(L_r)},$$

by convention

$$e^x = 1 + x + \frac{x^2}{2} + \ldots = \sum_{i=0}^{N} \frac{x^i}{i!}.$$

where we assume $x^{N+1} = 0$, $x^{N+2} = 0$, ...

$ch(\mathcal{E})$ is defined similarly to $c(\mathcal{E})$ but instead of e_i we use

$$e^{x_1} + \ldots + e^{x_r} = \sum_{k=0}^{N} \left(\frac{x_1^k}{k!} + \ldots + \frac{x_r^k}{k!}\right) =$$

$$= \sum_{k=0}^{N} \frac{1}{k!} p_k(x_1, \ldots, x_r), \text{ where}$$

$$p_k(x_1, \ldots, x_r) = x_1^k + \ldots + x_r^k.$$

<u>Remark</u> $c(\mathcal{E} \oplus \mathcal{E}') = c(\mathcal{E})\, c(\mathcal{E}')$,

but $ch(\mathcal{E} \oplus \mathcal{E}') = ch(\mathcal{E}) + ch(\mathcal{E}')$.

$$\Downarrow$$

<u>ch</u> is sometimes better than c.

But the information contained in c is the same as in ch.

$$ch(\mathcal{E}) = \sum_k \frac{1}{k!} p_k(x_1, \ldots, x_r) \Big/_{x_i = c_1(L_i)}$$

$$c(\mathcal{E}) = \sum_k e_k(x_1, \ldots, x_r) \Big/_{x_i = c_1(L_i)}$$

because of the Newton identities $\begin{pmatrix} p_k = f_k(e_1, \ldots, e_k) \\ e_k = g_k(p_1, \ldots, p_k) \end{pmatrix}$.

<u>Ex</u>: $p_1 = e_1$

$$e_2 = \frac{1}{2}\Big(\underset{\sum x_i^2 + 2\sum_{i<j} x_i x_j}{\overset{\parallel}{p_1^2}} - p_2\Big).$$

$$p_2 = p_1^2 - 2e_2.$$

<u>Remark</u>

Consider additive invariants $f(\mathcal{E})$ satisfying $f(\mathcal{E} \oplus \mathcal{E}') = f(\mathcal{E}) + f(\mathcal{E}')$.

$$f(\mathcal{E}) = \underset{H^0(X)}{\overset{\cap}{f_0(\mathcal{E})}} + \underset{H^2(X)}{\overset{\cap}{f_1(\mathcal{E})}} + \ldots$$

$$\mathcal{E} = \bigoplus_{i=1}^{r} L_i \Rightarrow f(\mathcal{E}) = \sum f(\underbrace{L_i}_{\text{line bundle}})$$

Assume $f(\underbrace{L}_{\text{line bundle}}) = a_0 + a_1 c_1(L) + a_2 c_1(L)^2 + \ldots$

any $a_0, a_1, a_2, \ldots$ give $f \sim$ additive invariant.

But ch satisfies another property:

$$ch(\mathcal{E} \otimes \mathcal{E}') = ch(\mathcal{E})\, ch(\mathcal{E}').$$

Then ch is unique (up to some rescaling).

Having introduced ch, what is Td?

Consider the expansion

$$B(x) = \frac{x}{1-e^{-x}} = \frac{x}{1-1+x-x^2/2+x^3/6+\dots}$$

$$= \frac{1}{1-x/2+x^2/6+\dots} = \frac{1}{1-\left(\frac{x}{2}-\frac{x^2}{6}+\dots\right)}$$

$$= 1+\left(\frac{x}{2}-\frac{x^2}{6}+\dots\right)+\left(\frac{x}{2}-\frac{x^2}{6}+\dots\right)^2$$

$$= 1+\frac{x}{2}+\frac{x^2}{12}-\frac{x^4}{720}+\frac{x^6}{30240}\dots$$

$$\frac{1}{4}-\frac{1}{6}=\frac{1}{12}$$

$$\frac{1}{8}-\frac{2}{2\cdot 6}+\frac{1}{24}=\frac{3-4+1}{24}=0$$

coefficients of $B(x)$ ~ Bernoulli numbers ~ Riemann zeta

$$B(x) = 1+\frac{x}{2}+\frac{x^2}{12}+\dots$$

If $\mathcal{E} = \oplus L_i$ then

<u>Todd character:</u>

$$Td(\mathcal{E}) := \prod B(c_1(L_i)).$$

(not additive like ch, but multiplicative like c).

<u>Remark</u>

The information contained in $Td(\mathcal{E})$ can be deduced from $c(\mathcal{E})$:

$$B(x_1)B(x_2)\dots B(x_r) = 1+e_1(x_1,\dots,x_r) + \left(\frac{1}{12}p_2+\frac{1}{2}e_2\right)+\dots$$

Notice that any polynomial $p(x_1,\dots,x_r)$ which is symmetric can be written as

$$p(x_1,\dots,x_r) = f(e_1(x_1,\dots,x_r),e_2,e_3,\dots).$$

So say on a computer we can compute

$$B(x_1)\dots B(x_r) = 1+f_1+f_2+\dots$$

$$f_i = f_i(e_1(x_1,\dots,x_r),e_2(x_1,\dots,x_r),\dots)$$

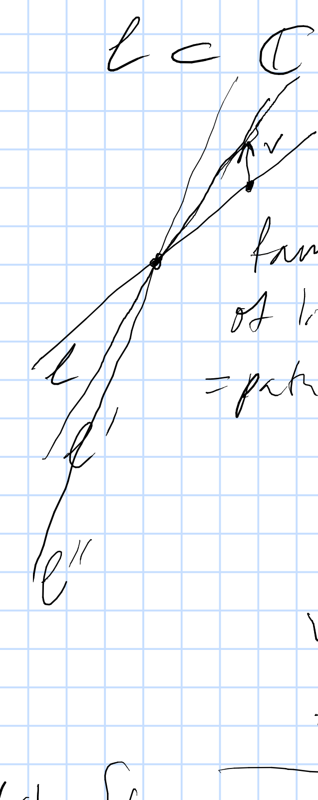
What is Td_X for a space X?

This is $Td(T_X)$ — tangent bundle of X.

Main example $X=\mathbb{C}P^n$.

How to understand T_X in terms of line bundles?

A point on $\mathbb{C}P^n$ is $\ell \subset \mathbb{C}^{n+1}$, $\dim \ell = 1$, $\ell \ni \{0\}$.



family of lines = path on $\mathbb{C}P^n$.

if $x\in\ell$, $x\neq 0$

$\ell' = (x+v)$.

$\Updownarrow$

$v:\ell\to\mathbb{C}^{n+1}$ linear map

$\ell' = x+v(x)$ $\forall x\in\ell$.

(*) $\{\text{linear maps } \ell\to\mathbb{C}^{n+1}\}\to T_\ell$ — tangent space at ℓ.

Clearly surjective.

not iso

Ker = maps $\ell\to\ell$.

$x\to\lambda x$ some $\lambda\in\mathbb{C}$.

This was for a fixed $\ell\in\mathbb{C}P^n$.

globally we have

$$\{\text{linear maps } \ell\to\mathbb{C}^{n+1}\} = \{\text{maps } \underbrace{(\ell\to\mathbb{C},\ell\to\mathbb{C},\dots,\ell\to\mathbb{C})}_{n+1}\}$$

is a fiber at ℓ of $(\text{dual tautological})^{\oplus n+1}$

$$= \underbrace{\mathcal{O}(1)\oplus\dots\oplus\mathcal{O}(1)}_{n+1}.$$

(*) becomes

$$\underbrace{\mathcal{O}(1)\oplus\dots\oplus\mathcal{O}(1)}_{n+1}\to T_{\mathbb{C}P^n}$$

surjective, ker = $\mathcal{O}(0) =: \mathcal{O}$. trivial line bundle.

Obtain s.e.s.

$$0\to\mathcal{O}\to\mathcal{O}(1)^{\oplus n+1}\to T_{\mathbb{C}P^n}\to 0.$$

$$\Rightarrow \underbrace{Td(\mathcal{O})}_{=1}\,Td(T_{\mathbb{C}P^n}) = B(c_1(\mathcal{O}(1)))^{n+1}$$

$c_1(\mathcal{O}(1)) = [H]\in H^2$ — hyperplane

$$\Rightarrow Td_X = Td(T_{\mathbb{C}P^n}) = B([H])^{n+1}.$$

HRR: for $\mathcal{O}(m)$

$$\int_{[X]} \underbrace{B([H])^{n+1}}_{=Td_X} e^{m[H]}$$

take H^{2n}-component, pair it with the fund. class in H_{2n}.

for $\mathbb{C}P^n$ means take the coefficient of $[H]^n$.

denote $[H]=x$ the formula becomes

$$\left(\left(\frac{x}{1-e^{-x}}\right)^{n+1}e^{mx}\right)_{\text{take the coeff of } x^n} = \binom{n+m}{m}.$$

Homework: check this formula explicitly.