<u>Statement</u> a cubic surface contains <u>27</u> lines.

<u>Main topic</u>:

<u>Grassmannian</u>

for r, n $Gr(n,r)$ as a set = the set of r-dimensional linear subspaces in $\mathbb{C}^n$.

equivalently is the set of $r-1$ dimensional projective subspaces in the $n-1$ dimensional projective space.

So for example projective lines in $\mathbb{C}P^3$ are parametrized by $Gr(4,2)$.

Another example $\mathbb{C}P^n = Gr(n+1,1)$.

<u>Plan</u>

1) Understand the geometry
2) compute cohomology
3) construct interesting vector bundles
4) Convert counting questions to computations with Chern classes
5) Get identities from Riemann-Roch.

1) Geometry of $Gr(n,r)$.

Cell decomposition:

Suppose $V \subset \mathbb{C}^n$, $\dim V = r$.

Pick a basis of V = r vectors $v_1, \ldots, v_r \in \mathbb{C}^n$.

Conversely, any collection $v_1, \ldots, v_r$ linearly indep. gives such V

$V = \mathrm{Span}(v_1, \ldots, v_r)$.

<u>Problem</u> many ways to choose a basis

$$(v_1, \ldots, v_r) \xrightarrow[\text{many to } 1]{} V$$

We want $1 \leftrightarrow 1$ correspondence.

To do it Let v_1 be such that

$$v_1 = (v_{11}, v_{12}, \ldots, 1, \underbrace{0, \ldots, 0}_{\text{as many zeros as possible}})$$

so $v_{1k_1} = 1$, $v_{1i} = 0$, $i > k_1$

k_1 is as small as possible.

What about v_2?

$v_2 \neq \lambda v_1$ $(\lambda \in \mathbb{C})$

also choose v_2 s.t.

$v_{2k_2} = 1$, $v_{2i} = 0$, $i > k_2$

Of course, $k_2 > k_1$. (otherwise v_2 would be v_1).

So on. produce $v_3, \ldots$

is v_2 unique?

v_2, v_2' $v_2 - v_2'$ has more zeroes in the end

$v_2 - v_2'$ must be proportional to v_1.

Let us assume further that

$$v_{2k_1} = 0.$$

Now v_2 is unique. Proceeding this way we obtain a sequence

$v_1, \ldots, v_r \in \mathbb{C}^n$, and a sequence $k_1, \ldots, k_r \in \mathbb{Z}$. s.t.

$v_{ik_i} = 1$, $v_{ij} = 0$ $(j > k_i)$

$v_{ik_j} = 0$ $(j < i)$.

<u>Claim</u> a collection satisfying these assumptions is unique.

<u>Proof</u> Homework.

$V \to$ $v_1, \ldots, v_r$, $k_1, \ldots, k_r$ canonical basis.

For each increasing sequence

$1 \le k_1 < k_2 < \ldots < k_r \le n$

Let $\sigma_{k_1 \ldots k_r} \subset Gr(n,r)$ be the set of subspaces V which produce $v_1, \ldots, v_r$, $k_1, \ldots, k_r$ with given $k_1, \ldots, k_r$ by the above construction.

How does $\sigma_{k_1, \ldots, k_r}$ look like?

$\sigma_{k_1, \ldots, k_r}$ looks like $\mathbb{C}^{\#\text{ of free coordinates}}$

$$\parallel$$

$$\mathbb{C}^{N_{k_1, \ldots, k_r}}$$

$N_{k_1, \ldots, k_r} = \#(i,j) \mid v_{ij}$ is allowed to take any value

$\Leftrightarrow$ $j < k_i$ $j \neq k_{i'}$, $i' < i$.

$$N_{k_1, \ldots, k_r} = \sum_{i=1}^{r} (k_i - 1) - (i-1) = \sum_{i=1}^{r} k_i - i.$$

<u>Examples</u>

1) $r = 1$ we have $*$ = anything

$\sigma_1 = \{(1, 0, \ldots, 0)\}$ $k_1 = 1$

$\sigma_2 = \{(*, 1, 0, \ldots, 0)\}$ $k_1 = 2$

$\sigma_3 = \{(*, *, 1, 0, \ldots 0)\}$ $k_1 = 3$

2) $r = 2$ $n = 4$

$(k_1, k_2) = (1,2), (1,3), (1,4), (2,3), (2,4), (3,4)$

σ_{k_1, k_2} $(1,2)$: $v_1 = (1,0,0,0)$, $v_2 = (0,1,0,0)$ only one V

$(1,3)$: $v_1 = (1,0,0,0)$, $v_2 = (0,*,1,0)$ $(*)$

$\left.\begin{array}{l} \dim V = 2, \ V \subset \mathrm{span}(e_1, e_2, e_3) \\ V \ni e_1 \end{array}\right\} \overset{?}{\Longrightarrow}$ V is spanned by v_1, v_2 as in $(*)$

no.

but $\Rightarrow$ $\sigma_{1,2} \cup \sigma_{1,3}$.

$(1,4)$: $v_1 = (1,0,0,0)$

$v_2 = (0,*,*,1)$

$\dim V = 2$ $V \ni e_1 \Rightarrow \sigma_{1,2} \cup \sigma_{1,3} \cup \sigma_{1,4}$.

$(2,3)$: $v_1 = (*,1,0,0)$

$v_2 = (*,0,1,0)$

$\dim V = 2$, $V \subset \mathrm{span}(e_1, e_2, e_3) \Rightarrow$

$\sigma_{2,3} \cup \sigma_{1,2} \cup \sigma_{1,3}$.

In general $\sigma_{k_1, \ldots, k_r}$ can be described by putting conditions on V, and removing other σ-s.

$(2,4)$ $v_1 = (*,1,0,0)$

$v_2 = (*,0,*,1)$

$(3,4)$ $v_1 = (*,*,1,0)$

$v_2 = (*,*,0,1)$.

denote $\underline{k} = (k_1, \dots, k_r)$

$$I_{n,r} = \{ \overline{k} \mid 1 \le k_1 < \dots < k_r \le n \}$$

then $$Gr(n,r) = \bigcup_{\underline{k} \in I_{n,r}} \sigma_{\underline{k}}$$

$\sigma_{\underline{k}}$ are called <u>cells</u>.

(Schubert cells).

How to put a structure of a manifold on $Gr(n,r)$?

The <u>big cell</u> is $\sigma_{\underline{k}}$ $\underline{k} = (n-r+1, \dots, n)$.

equivalently $k_1 = n-r+1$, equivalently

$$V \in \text{big cell} \iff V \cap \mathrm{Span}(e_1, \dots, e_{n-r}) = \{0\}.$$

<u>Prop</u>

For any V there exist a basis of $\mathbb{C}^n$ w.r.t which V is in the big cell.

Proof clear.

So big cells for different choices of basis in $\mathbb{C}^n$ cover $Gr(n,r)$, we use big cells to construct charts on $Gr(n,r)$.

(remember cells look like affine spaces

$$\text{big cell} \cong \mathbb{C}^N \cong \mathbb{C}^{(n-r)r} \cong \mathbb{R}^{2(n-r)r}.$$

$$N = (n-r+1-1) + (n-r+2-2) + \dots = (n-r) \cdot r$$

Cohomology of $Gr(n,r)$.

Prop $\overline{\sigma_{\underline{d}}} = \bigcup_{\underline{d}' \leq \underline{d}} \sigma_{\underline{d}'}$ for certain order $\leq$ on $I_{n,r}$.

Proof

if $V \in \sigma_{\underline{d}}$ $\Rightarrow$

$$\dim(V \cap \operatorname{span}(e_1, \dots, e_{d_1})) \geq 1$$
$$\dim(V \cap \operatorname{span}(e_1, \dots, e_{d_2})) \geq 2 \quad (*)$$
$$\vdots$$
$$\dim(V \cap \operatorname{span}(e_1, \dots, e_{d_r})) \geq r.$$

$\overline{\sigma_{\underline{d}}} = \{V \mid \text{satisfying } (*)\}$.

Suppose conversely V satisfies $(*)$. $V \in \sigma_{\underline{d}'}$ then:

$d_1' \leq d_1$

$d_2' \leq d_2$

$\vdots$

So for the order $\underline{d}' \leq \underline{d}$ if $d_i' \leq d_i$ the statements holds. $\lrcorner$

(Let's list all the cells $\sigma_{\underline{d}_1}, \dots, \sigma_{\underline{d}_m}$ so that for $i<j$ then $\underline{d}_i < \underline{d}_j$.

Let $X_i = \bigcup_{i \leq i} \sigma_{\underline{d}_i}$.

$X_1 \subset X_2 \subset \dots \subset X_m = Gr(n,r)$.

X_i / X_{i-1} = Sphere of dimension $2(\sum d_j - j)$ where $\underline{d} = \underline{d}_i$.

Hence $H^k(X_i, X_{i-1}, \mathbb{Z}) = \begin{cases} 0 & k \neq 2(\sum d_j - j) \\ \mathbb{Z} & \text{otherwise} \end{cases}$

by long exact cohomology sequence

$2(\sum d_j - j)$ is even!

we get:

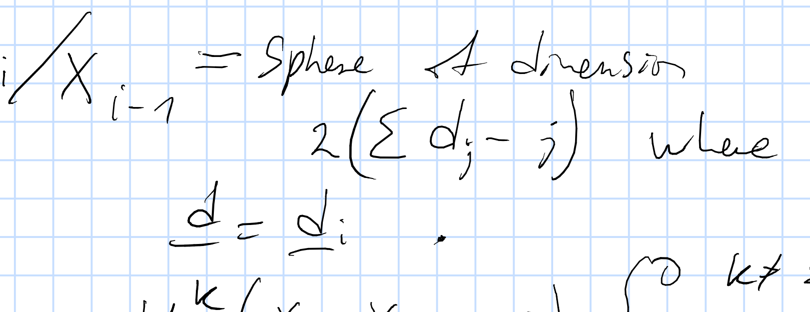
$$H^k(Gr(n,r), \mathbb{Z}) = \begin{cases} 0 & k \text{ is odd} \\ \mathbb{Z}^{\#\underline{d} :\, N_{\underline{d}} = k/2} & k \text{ even} \end{cases}$$

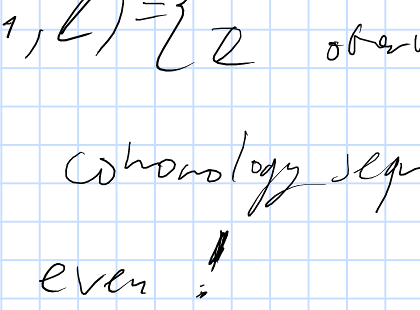
Ex $I_{4,2}$:

$\sigma_{1,2}, \sigma_{1,3}, \sigma_{1,4}, \sigma_{2,3}, \sigma_{2,4}, \sigma_{3,4}$.

X_1 X_2 X_3 X_4

$\overline{\sigma_{2,3}} = \sigma_{2,3} \cup \sigma_{1,3} \cup \sigma_{1,2}$ (don't have $\sigma_{1,4}$)

$X_4 =$ 

X_4 / X_3 

$Gr(4,2)$:

	$\sigma_{3,4}$	$H^8 = \mathbb{Z}$
	$\sigma_{2,4}$	$H^6 = \mathbb{Z}$
planes	$\sigma_{1,4}$ $\sigma_{2,3}$	$H^4 = \mathbb{Z}^2$
lines	$\sigma_{1,3}$	$H^2 = \mathbb{Z}$
point	$\sigma_{1,2}$	$H^0 = \mathbb{Z}$

Next time: Vectorbundles, chern classes, ...