Today: Chern classes + ..

Before: # solutions of a system of equations of degrees $d_1, d_2, \dots, d_m$ is $d_1 d_2 \dots d_m$.

Based on: Intersecting subsets $\longleftrightarrow$ cup product in cohomology.

Dold's book: most complete

Chriss, Ginzburg more manageable approach
2.6 Borel-Moore Homology
probably the best way to compare intersections with cup products.

Want:
M oriented manifold (maybe not compact) dim n
$L \subset M$ oriented submanifold of dimension d
$d' = n-d$ codimension

there is a class in $H^{d'}(M, M-L)$.

Recall $H^i(M, M-L)$ depends only on the neighborhood of L in M by excision.
$M \supset U \supset L$ U open then $H^i(M, M-L) \to H^i(U, U-L)$ is an isomorphism.

$H^i(M, M-L)$ doesn't depend on M.

Prop $H^i(M, M-L) = \begin{cases} 0 & i < d' \\ \mathbb{Z}^{\#\text{of connected components of } L} & i = d'. \end{cases}$

choice of orientation $\to$ $c_L \in H^{d'_L}(M, M-L)$.

$L_1, L_2 \subset M$ oriented manifolds of codim d'_1, d'_2

$c_{L_1} \cap c_{L_2} \in H^{d'_1 + d'_2}(M, (M-L_1) \cup (M-L_2)) = H^{d'_1+d'_2}(M, M-(L_1 \cap L_2))$

$c_L \in \underline{H^{d'}(M, M-L)}$
chains on $M \to \mathbb{Z}$
chains on $M-L \to 0$

if $L_1 \cap L_2$ is a manifold, L_1 and L_2 locally around $L_1 \cap L_2$ look like
$L_1 = \mathbb{R}^{n-d'_1-d'_2} \times \mathbb{R}^{d'_2} \times \{0\} \subset \mathbb{R}^n$
$L_2 = \mathbb{R}^{n-d'_1-d'_2} \times \{0\} \times \mathbb{R}^{d'_1} \subset \mathbb{R}^n$
$L_1 \cap L_2 = \mathbb{R}^{n-d'_1-d'_2} \times \{0\} \times \{0\} \subset \mathbb{R}^n$.

locally $c_{L_1 \cap L_2} = c_{L_1} \cup c_{L_2}$.
i.e. holds when restricted to open sets of some cover.
$\rightsquigarrow$ globally $c_{L_1 \cap L_2} = c_{L_1} \cup c_{L_2}$

$H^i(M, M-L) \to H^i(M) \to H^i(M-L)$

Send $c_{L_1}, c_{L_2}, c_{L_1 \cap L_2}$ to $H^*(M)$ get the result:

class of $L_1 \cap L_2$ in $H^{d'_1+d'_2}(M) =$ cup product of the classes of L_1, L_2.

modifying L by subsets of codimension ≥ 2 doesn't affect $H^{d'_L}(M, M-L)$.

Chern classes of complex vector bundles.

Axioms If $\mathcal{E} \to X$ is a complex vector bundle of rank r, then there exist elements $c_i(\mathcal{E}) \in H^{2i}(X, \mathbb{Z})$ $i=0,\dots,r$. Satisfying

0) $c_0(\mathcal{E}) = 1$

1) $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ exact sequence $\Rightarrow$

$$c_m(\mathcal{E}) = \sum_{i+j=m} c_i(\mathcal{E}_1) \cup c_j(\mathcal{E}_2).$$

2) Functoriality: Given a continuous map $f: X \to Y$, $\mathcal{E}, F$ vector bundles on X, Y resp. such that $\exists\ \hat{f}: \mathcal{E} \to F$ s.t.

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\hat{f}"] \arrow[d] & F \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutative, $\forall x \in X$ the restriction $\mathcal{E}_x \to F_{f(x)}$ is iso, (this is called a vector bundle map) then

$$c_i(\mathcal{E}) = f^* c_i(F). \quad f^*: H^i(Y) \to H^i(X).$$

3) If $\mathcal{L} \to X$ is a line bundle, then $c_1(\mathcal{L}) =$ Euler class of $\mathcal{L}$ (more on that later).

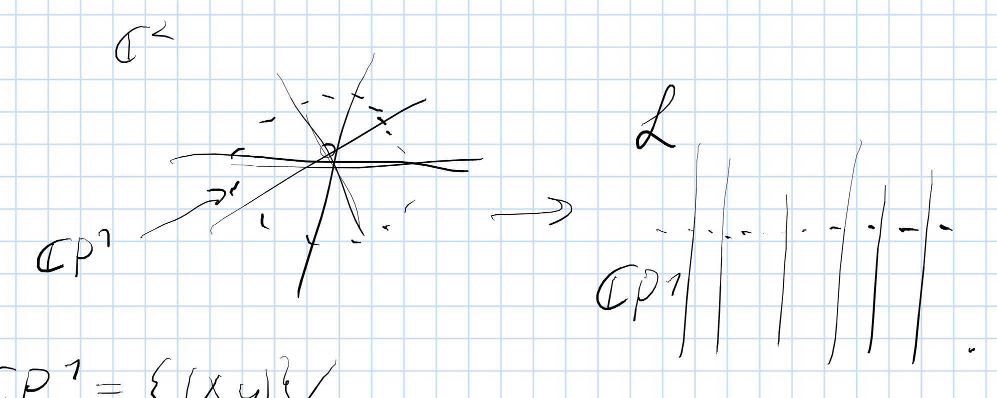
How to compute Chern classes?

Examples on $\mathbb{C}P^n$.

Tautological line bundle. (line $= \mathbb{C}$)

$\mathbb{C}P^n = \{(x_0,\dots,x_n) \neq 0\}/\sim \; = \{$lines (1-dim vector spaces) passing through the origin of $\mathbb{C}^{n+1}\}$

Let $\mathcal{L} = \{(\ell, x) \mid \ell \in \mathbb{C}P^n$, i.e. a line passing through 0. $x \in \ell\}$



__Charts__ $\mathbb{C}P^1 = \{(x,y)\}/\sim$

2 charts $U_1: y \neq 0 \xrightarrow{\text{make } y=1} x \in \mathbb{C}$ $\quad U_1 \xrightarrow{\sim} \mathbb{C}$

$U_2: x \neq 0 \xrightarrow{\text{make } x=1} y \in \mathbb{C}$ $\quad U_2 \xrightarrow{\sim} \mathbb{C}$.

On $U_1 \cap U_2$ $\quad x \neq 0, y \neq 0$

on U_1: $(x, 1) \sim (1, x^{-1})$

on U_2 $\quad (1, x^{-1})$

$x \in U_1 \supset U_1 \cap U_2 \subset U_2 \ni x^{-1}$.

Example 1) __Tautological bundle__

on U_1 line is given by $(x, 1)$; point on the line for $(x,1)$ is (a,b) s.t. (a,b) is parallel to $(x,1)$

$a = bx$

$\mathcal{L}/_{U_1} = \{(x, b) \mid x \in \mathbb{C}, b \in \mathbb{C}\}$.

Similarly on U_2 line $(1, y)$; point on line: (a, b) : $b = ya$.

On the intersection: (x, b) on $\mathcal{L}/_{U_1}$ $\quad x \in U_1 \cap U_2$

$(x,1)$, $\; (bx, b) \in \mathcal{L}$

$(1, x^{-1}) \quad (bx, b)$

(x^{-1}, bx) on $\mathcal{L}/_{U_2}$.

$\mathcal{L} = \underset{(x,b)}{\mathbb{C} \times \mathbb{C}} \cup \underset{(y,a)}{\mathbb{C} \times \mathbb{C}}$ glued along $(x, b) \to (x^{-1}, bx)$

Q) Does $\mathcal{L}$ have non-zero cross-sections?

map $\mathbb{C}P^1 \to \mathcal{L}$ s.t. $\mathbb{C}P^1 \to \mathcal{L} \to \mathbb{C}P^1$ is the identity.

Do we have sections given by rational functions?

on U_1 $\quad b = f_1(x)$ $\quad f_1$ rational function, well defined for all $x \in \mathbb{C}$. So $f_1 = b_0 + b_1 x + \dots + b_n x^n$.

on U_2 $\quad a = f_2(y)$ $\quad$ Similarly $f_2 = a_0 + a_1 y + \dots + a_m y^m$.

$x \neq 0$: $\quad \underset{a}{f_2(x^{-1})} = \underset{b}{f_1(x)}\, x \quad \left(\begin{matrix} y = x^{-1} \\ a = bx \end{matrix}\right)$.

$$a_0 + a_1 x^{-1} + \dots + a_m x^{-m} = b_0 x + b_1 x^2 + \dots + b_n x^{n+1}.$$

not possible.

We will see: if $\mathcal{L}$ has a holomorphic section f $\Rightarrow$ $c_1(\mathcal{L}) = [Z]$ where $Z = \{x \in X \mid f(x) = 0\}$.

__Example 2__ Dual tautological bundle.

$\mathcal{L}^* = \{\ell \in \mathbb{C}P^1$, linear map $\ell \to \mathbb{C}\}$.

Computing the transition function between U_1 and U_2

on U_1 $\quad (x, 1)$ $\quad \forall \lambda \in \mathbb{C}$ we have a linear map $\quad (a, b) \to \lambda b$ $\quad (a = bx)$

on U_2 $\quad (1, y)$ $\quad \forall \mu \in \mathbb{C}$ we have a linear map

$(a, b) \to \mu a$.

$b = ay$

on $U_1 \cap U_2$ $\quad x \in U_1 \quad x \neq 0 \quad (a, b) \to \lambda b$

go to U_2 $\quad (a, b) \to \lambda b = \lambda a y = \underset{\mu}{(\lambda x^{-1})}\, a$

$\mathcal{L}^*$ is obtained by gluing

$\underset{(x,\lambda)}{\mathbb{C}^2}$ and $\underset{(y,\mu)}{\mathbb{C}^2}$

$(x, \lambda) \to (x^{-1}, \lambda x^{-1})$

Compare with $x, b \to (x^{-1}, bx)$.

Holomorphic Section:

$$\lambda = \lambda_0 + \lambda_1 x + \dots + \lambda_n x^n$$

$$\mu = \mu_0 + \mu_1 y + \dots + \mu_m y^m$$

$\lambda x^{-1} = \mu$:

$$\lambda_0 + \lambda_1 x + \dots + \lambda_n x^n = (\mu_0 + \mu_1 x^{-1} + \dots + \mu_m x^{-m})x$$

Notice: $\underbrace{\lambda_0 + \lambda_1 x + \dots + \lambda_n x^n}_{1, x, \dots, x^n}$ $\quad (x, 1, x^{-1}, \dots)$

All sections are given by $\quad \mu_0 = \lambda_1 \quad \mu_1 = \lambda_0$

$$\lambda_0 + \lambda_1 x = (\mu_0 + \mu_1 x^{-1})x.$$

Let us compute the zero set of a section.

$(\lambda_0, \lambda_1) \neq (0,0)$ $\quad$ Cases: 1) $\lambda_1 \neq 0$: $\quad x = -\frac{\lambda_0}{\lambda_1} \in U_1$ is the unique zero point

(on U_2 $\; y = -\frac{\lambda_1}{\lambda_0}$, if $\lambda_0 \neq 0$ we get a point, this is $U_1 \cap U_2$).

2) $\lambda_1 = 0$ $\quad$ no points on U_1 where the section is 0.

On U_2: $\quad y = 0$ is a single point.

__Guess__ the class of the zero set of a section. Depends only on bundle, not on the section.

This is called the Euler class. $e(\mathcal{L})$.

See 3) of the Axioms.