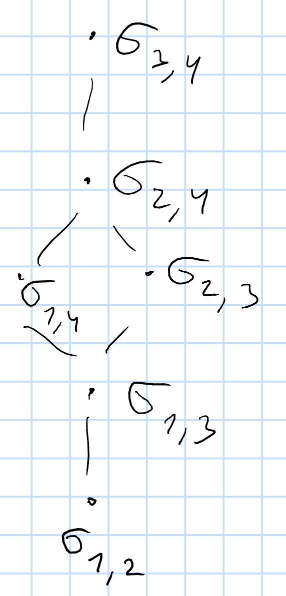


$Gr(2,4)$

$\sigma_{k_1 \dots k_r} \rightarrow$ subspaces are spanned by $v_1, \dots, v_r$

$$v_i = (\dots * \dots * 0 * * 1 \quad 0 \dots 0)$$
$$\qquad\quad k_{i-1} \qquad k_i$$

A ***partition*** is a non-increasing sequence of integer numbers $\lambda_1 \geq \lambda_2 \geq \dots \lambda_e \geq \dots$ such that $\lambda_e = \lambda_{e+1} = \dots = 0$ some e.

A partition of n is a partition s.t. $\sum_{i=1}^{\infty} \lambda_i = n$.

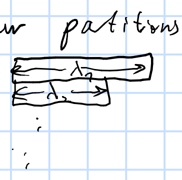
Partitions of 5: 5, 4+1, 3+1+1, 3+2, 2+2+1, 2+1+1+1, 1+1+1+1+1

5, 0, 0, … , 0, …

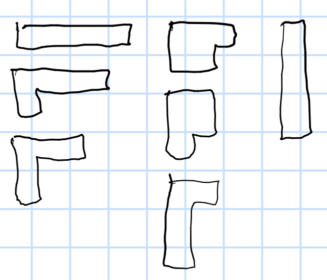
4, 1, 0, … , 0

3, 1, 1, 0, 0 …

Draw partitions as Young diagrams:



Partitions of 5:



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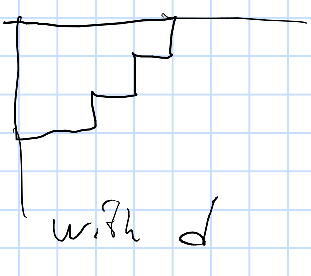
cells $\rightarrow$ Young diagrams

the largest cell $\rightarrow$ class in $H^0 \rightarrow$.

arbitrary cell $\rightarrow$ Young diagram with d boxes, i.e. Partition of d.

Complex dimension $r(n-r) - d$



$$1 \leq k_1 < k_2 < \dots < k_r \leq n$$

$$n-r+1, \dots, n \rightarrow 0, 0, 0$$

general:

$$n-r+1-k_1,\ n-r+2-k_2, \dots, n-k_r.$$

Prop this is a partition:

$$\underset{\lambda_1}{\underset{\shortparallel}{n-r+1-k_1}} - \left(\underset{\lambda_2}{\underset{\shortparallel}{n-r+2-k_2}}\right) = k_2 - k_1 - 1 \geq 0$$

Similarly we obtain

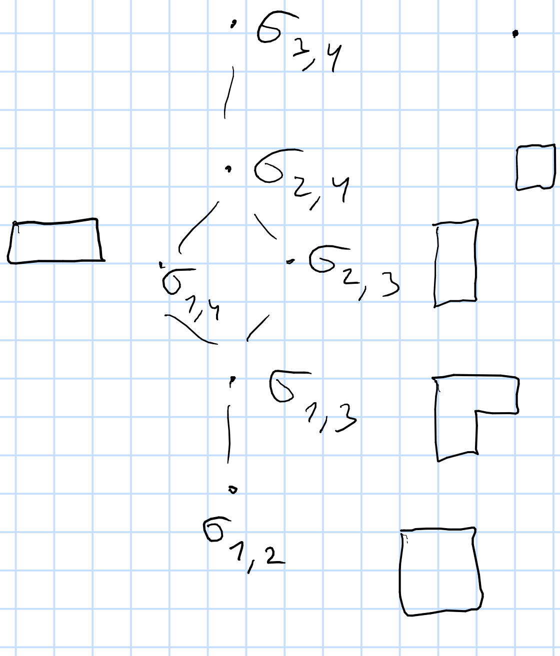
$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0 \geq 0 \dots$$

The size of λ (= # of boxes)

$$= \sum \lambda_i = \sum_{i=1}^{r} (n-r+i-k_i) =$$

$$= r(n-r) - \underbrace{\sum_{i=1}^{r} (k_i - i)}_{\text{complex dimension of the cell.}}$$

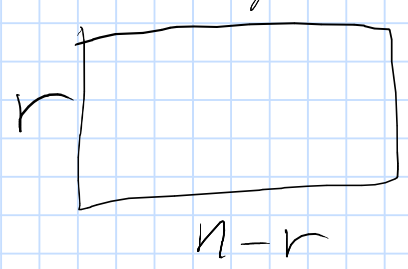
Example:



Note that not all partitions appear: $\lambda_1 = n-r+1-k_1 \leq n-r$ so the width of the diagram is $\leq n-r$.

We have at most r positive numbers $\Rightarrow$ the height of the diagram is $\leq r$.

So cells are in bijection with diagrams which fit inside



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The "row" cells:

$\underbrace{\boxed{}}_{k}$ $\qquad \lambda_1 = k \quad \lambda_2 = \dots = 0$

$$k_1 = n-r+1-k$$
$$k_2 = n-r+2$$
$$\vdots$$

$k=3$

$$\begin{array}{l} v_1 = \\ v_2 = \\ v_3 = \\ \\ \\ \end{array} \begin{array}{cccccccccccccc} * & \dots & * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \dots & * & * & 0 & * & * & * & 1 & 0 & 0 & 0 \\ * & \dots & * & * & 0 & * & * & * & 0 & 1 & 0 & 0 \\ * & \dots & * & * & 0 & * & * & * & 0 & 0 & 1 & 0 \\ * & & * & * & 0 & * & * & * & 0 & 0 & 0 & 1 \end{array}$$

$\overline{\sigma_\cdot} = \Big(V \subset \mathbb{C}^n$ s.t. $V \cap \operatorname{span}(e_1, \dots, e_{n-r+1-k}) \neq \{0\}$

So think $\operatorname{span}(e_1, \dots, e_{n-r+1-k}) = W$

W is fixed

$$\overline{\sigma_{k_1 \dots}} = \{ V \subset \mathbb{C}^n \mid V \cap W \neq \{0\} \}$$

The "column" cells

$$\lambda = (\underbrace{1, 1, \dots, 1}_{k}, 0, \dots 0)$$

$$k_1 = n-r$$
$$k_2 = n-r+1$$
$$\vdots \qquad \vdots$$
$$k_k = n-r+k-1$$
$$k_{k+1} = n-r+k+1$$
$$\vdots$$
$$k_r = n$$

$$v_1 = \left(\begin{array}{cccccccccccc} \dots & * & * & * & * & 1 & 0 & 0 & \dots & 0 & \cdot \\ & * & * & * & * & 0 & 1 & 0 & 0 & 0 \\ \dots & * & * & * & 0 & 0 & 1 & 0 & 0 & 0 \\ \dots & * & * & * & 0 & 0 & 0 & * & 1 \end{array} \right) \Big\} k$$
$$\vdots$$

$$v_1, \dots, v_k \subset \underbrace{\text{Span of } e_1, \dots, e_{n-r+k-1}}_{W}$$

$$\dim(W \cap V) \geq k.$$

"Usual dimension of $W \cap V$" is

$$n - (n - \dim V) - (n - \dim W) =$$
$$= \dim V + \dim W - n$$
$$= k-1.$$

So the "column condition" which cuts out the closure of the column cell is

$$\dim(W \cap V) > k-1.$$

Vector bundles on $Gr(r,n)$.

a point x on $Gr(r,n)$ corresponds to a subspace $V_x \subset \mathbb{C}^n$, so we have a vector bundle on $Gr(r,n)$ whose fiber over x is V_x.

(it is a subset of $Gr(r,n) \times \mathbb{C}^n$ }

$$\| $$

$$\{x \in Gr(r,n),\ y \in \mathbb{C}^n \mid y \in V_x\}.$$

Denote this bundle by E.
Another vector bundle is such that fiber over $x \in Gr(r,n)$ is the quotient space $\mathbb{C}^n/V_x$.
Call it E'.
So we have a short exact sequence of vector bundles

$$0 \to E \to \underbrace{\mathbb{C}^n \times Gr(r,n)}_{\text{trivial bundle}} \to E' \to 0.$$

Consider Chern classes

$c_i(E)$ $\quad c_i(E')$.

We have:
$\forall k > 0$

$$\sum_{i+j=k} c_i(E)\, c_j(E') = 0$$

(because of the s.e.s)

$Gr(2,4)$ $\quad$ Let $c_i(E) = a_i$
$\quad c_i(E') = b_i$

a_1, a_2, b_1, b_2

$k=1$: $a_1 + b_1 = 0 \Rightarrow b_1 = -a_1$
$k=2$: $a_2 - a_1^2 + b_2 = 0 \Rightarrow b_2 = a_1^2 - a_2$

$k=3$: $a_1 b_2 + a_2 b_1 = 0 \Rightarrow$

$$0 = a_1(a_1^2 - a_2) - a_1 a_2 = a_1^3 - 2a_1 a_2$$

$k=4$: $a_2 b_2 = 0$: $\quad a_2(a_1^2 - a_2) = 0$

So:

$$\begin{cases} 2a_1 a_2 = a_1^3 & (\mathrm{I}) \\ a_2^2 = a_1^2 a_2 & (\mathrm{II}) \end{cases}$$

Combine (I, II) (see Gröbner basis)

$$(\mathrm{I})\, a_2 - 2(\mathrm{II}) \cdot a_1 = a_1^3 a_2 - 2a_1^3 a_2$$
$$= -a_1^3 a_2 = 0$$

Substitute in (I):

$$a_1^5 = 0$$

Basis: $1, a_1, a_1^2, a_1^3, a_1^4,$
a_2, everything else expressed in terms of these.

$6 = \dim H^*(Gr(2,4))$,

maybe $H^*(Gr(2,4)) = \mathbb{Q}[a_1, a_2, b_1, b_2] / \text{equations}$

Maybe more generally:

$$H^*(Gr(r,n)) = \mathbb{Q}[a_1, \ldots, a_r, b_1, \ldots, b_{n-r}] / \text{relations}$$

$$\forall k: \quad \sum_{i+j} a_i b_j = 0 \qquad (a_0 = b_0 = 1).$$

How to connect cells with vector bundles?

Look at row cells:
we have W
$\{V : V \cap W \neq \{0\}\}$.
Special case $\dim W = 1$
$W = \mathrm{span}(x)$ $\quad \{V \mid V \ni x\}$

$$\begin{array}{ccc} \mathbb{C}^n & \to & \mathbb{C}^n/V \\ \cup & & \cup \\ x & \to & 0 \end{array} \qquad \Updownarrow$$

$V \ni x \iff x$ goes to 0.

$x \in \mathbb{C}^n$, so it gives a section of the trivial bundle, we compose it with the map
$\mathbb{C}^n \times Gr(r,n) \to E'$ obtain a section of E', which is zero precisely when $V \ni x$

So the closure of the corresponding cell $\underbrace{\qquad\qquad}_{n-r} \to \lambda = (n-r, 0, 0, \ldots, 0)$ denote by $(n-r)$

Corollary $\quad e(E') = [\sigma_{(n-r)}] \in H^{2(n-r)}(Gr(r,n), \mathbb{Z})$.

$$\|$$
$$a_{n-r}$$
$$\|$$
$$ch_{n-r}(E') \qquad ⊣$$

What about column cells?

$\}r$ $\quad \dim W = n-1$ $\quad \{V \mid \dim V \cap W = r\}$
$\iff V \subset W$.

$$V_x \to \mathbb{C}^n \to \mathbb{C}^n/W \simeq \mathbb{C}$$

$V_x \subset W \iff$ this map is 0.
this map is a linear map $V_x \to \mathbb{C}$,
so is a vector in V_x^*.

So we have a section of E^*, which vanishes precisely when $V \subset W$.
So $\quad e(E^*) = [\sigma_{\underbrace{(1,\ldots,1)}_{r}}] \in H^{2r}$.

Question what is $c_i(E^*)$ in terms of $c_i(E)$?

$$c_i(E) = (-1)^i c_i(E^*).$$

Consider $E \to (-1)^i c_i(E^*)$.
We can check the axioms for Chern classes.
For line bundles on $\mathbb{P}^1$
$\mathcal{O}(1)^* = \mathcal{O}(-1)$.
$c_1(\mathcal{O}(-1)) = -c_1(\mathcal{O}(1))$.

Exact sequence:

$$0 \to \underbrace{\mathcal{O}(-1)}_{c_1} \to \mathbb{C}^2 \times \mathbb{P}^1 \to \underbrace{\mathcal{O}(1)}_{c_1} \to 0$$

$$+$$
$$\downarrow$$
$$0.$$

⊣

Corollary $\quad [\sigma_{\underbrace{(1,\ldots,1)}_{r}}] = (-1)^r a_r$.

What about other situations,
$\dim W =$ arbitrary.
next time:
similar:

$$c_i(E) = [\, r-i+1 \text{ sections become linearly dependent} \,].$$