Splitting principle

Thm Suppose X top space, $\mathcal{E} \to X$ complex vector bundle, let $\mathbb{P}(\mathcal{E})$ be the projectivization

(points of $\mathbb{P}(\mathcal{E})$ are pairs $x \in X$, $\ell \subset \mathcal{E}_x$

So when $\mathcal{E}$ is trivial $\mathcal{E} = X \times \mathbb{C}^n$ $\quad \mathbb{P}(\mathcal{E}) = X \times \mathbb{P}^{n-1}$.)

Consider $\mathbb{P}(\mathcal{E}) \to X$, the tautological line bundle $\tau_{\mathbb{P}(\mathcal{E})} \to \mathbb{P}(\mathcal{E})$, dual bundle bundle $\tau^*_{\mathbb{P}(\mathcal{E})} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

$\eta := c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \in H^2(\mathbb{P}(\mathcal{E}), \mathbb{Z})$.

$\pi : \mathbb{P}(\mathcal{E}) \to X$, so we obtain

$\pi^* : H^i(X) \to H^i(\mathbb{P}(\mathcal{E}))$.

$$H^i(X) \oplus H^{i-2}(X) \dots \oplus H^{i-2n+2}(X) \to H^i(\mathbb{P}(\mathcal{E}))$$

$$a_0, a_1, \dots, a_{n-1} \to a_0 + \eta a_1 + \dots + \eta^{n-1} a_{n-1}$$

this is an isomorphism.

So, $H(\mathbb{P}(\mathcal{E})) = H(X) \oplus \eta H(X) + \dots \eta^{n-1} H(X)$.

Corollary $\pi^* : H(X) \to H(\mathbb{P}(\mathcal{E}))$ is injective.

Pf The statement (the isomorphism) is true when $\mathcal{E}$ is trivial. by Künneth.

By Mayer–Vietoris + 5 lemma obtain the general statement.

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over $\mathbb{P}(\mathcal{E})$ we have exact sequence

$$0 \to \tau \to \pi^* \mathcal{E} \to \mathcal{E}' \to 0$$

rank $n-1$

Repeating this procedure for $\mathcal{E}'$

We obtain a sequence of projectivizations

$F(\mathcal{E}) \to \mathbb{P}(\mathcal{E}') \to \mathbb{P}(\mathcal{E}) \to X$

$\pi^* : F(\mathcal{E}) \to X$

over $x \in X$ we have the set of all complete flags in $\mathcal{E}_x$.

$H^*(X) \overset{\pi^*}{\subset} H^*(F(\mathcal{E}))$

$F(\mathcal{E})$ is called the complete relative flag variety.

over $F(\mathcal{E})$ $\quad \pi^*(\mathcal{E})$ has a filtration by subbundles

$F_1 \subset F_2 \subset \dots \subset F_{n-1} \subset F_n = \mathcal{E}$

So that $L_i = F_i / F_{i-1}$ are line bundles

By axioms of Chern classes

$$\pi^*(c(\mathcal{E})) = c(\pi^*(\mathcal{E})) =$$

$$= \prod_{i=1}^{n} c(L_i) = \prod_{i=1}^{n} (1 + c_1(L_i)) =$$

$$= \sum_{i=0}^{n} e_i(c_1(L_1), \dots, c_1(L_n)).$$

# of lines on cubic surface.
 $\quad$ $\uparrow$ $\mathbb{P}^3$

Setup

x, y, z, u coordinates on $\mathbb{P}^3$.

cubic surface is given by

$a_0 x^3 + a_1 y^3 + a_2 z^3 + a_3 u^3 + a_4 x^2 y + \dots$ $\quad (*)$ (many terms).

a line (a point in $Gr(2,4)$)

coordinates on line are s, t

$(x_1, x_2, x_3, x_4) s + (y_1, y_2, y_3, y_4) t$

$= (x_1 s + y_1 t,\ x_2 s + y_2 t, \dots)$

when is the line on the cubic?

plug $(x_1 s + y_1 t, \dots)$ into the equation $(*)$.

after opening brackets

$$f_0(x_1, x_2 \dots y_3, y_4) s^3 + f_1(\dots) s^2 t + f_2(\dots) s t^2 + f_3\, t^3. \quad (**)$$

if $f_0 = f_1 = f_2 = f_3 = 0$ then our line is on the surface.

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consider $E \to Gr(2,4)$ tautological bundle.

over $x \in Gr(2,4)$ the fiber of E, $E_x \subset \mathbb{C}^4$. $\dim E_x = 2$.

$(**)$ is a degree 3 polynomial function on E_x.

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degree n polynomials on V (V is a vector space) are given by elements of

$$Sym^n(V^*) = \underbrace{V^* \otimes \dots \otimes V^*}_{n} / \{ x_1 \dots \otimes x_i \otimes x_{i+1} \otimes \dots x_n - x_1 \otimes \dots x_{i+1} \otimes x_i \dots x_n \}.$$

(any degree n polynomial is a sum of products of n linear polynomials)

linear polynomials are elements of V^*.

Example V is 2-dimensional with basis e_1, e_2, the dual space has basis e^1, e^2

$e^1(x e_1 + y e_2) = x$

$e^2(x e_1 + y e_2) = y$

$V^* \otimes V^* \otimes V^*$ has basis

$e^1 \otimes e^1 \otimes e^1 \to x^3$

$e^1 \otimes e^1 \otimes e^2 \to x^2 y$

$e^1 \otimes e^2 \otimes e^1 \to x^2 y$

e^1

$\vdots$

8 in total

after identifying tensors up to permutations

$e^1 \otimes e^1 \otimes e^1 \quad x^3$

$e^1 \otimes e^1 \otimes e^2 \quad x^2 y$

$e^1 \otimes e^2 \otimes e^2 \quad x y^2$

$e^2 \otimes e^2 \otimes e^2 \quad y^3$.

Construct a vector bundle $Sym^3(E^*)$ over $Gr(2,4)$.

So $(**)$ gives a section S of $Sym^3(E^*)$, so if the intersection of S and the zero section is transversal, then

$$\# \text{ of } 0\text{-s} = e(Sym^3(E^*)).$$

$$\mathbb{Z} \cong H^8(Gr(2,4))$$

Splitting principle

$F(E^*) \to Gr(2,4)$

$H^*(F(E^*)) \supset H^*(Gr(2,4))$

over $F(E^*)$ we have

$0 \to L_1 \to E^* \to L_2 \to 0$

s.e.s. $\quad c_1(L_1) = x \quad c_1(L_2) = y$

$E^* = L_1 \oplus L_2$ (non-holomorphic)

$Sym^3(E^*) = L_1^3 \oplus L_1^2 L_2 \oplus L_1 L_2^2 \oplus L_2^3$.

$\qquad c_1 \downarrow$

$H^2(F(E^*)) \ni 3x \quad 2x+y \quad x+2y \quad 3y$.

$$\pi^* e(Sym^3(E^*)) = 3x(2x+y)(x+2y)3y.$$

$= 9(xy)(2x^2 + 5xy + 2y^2)$

$= 9(c_2)(2c_1^2 + c_2)$.

$(1+x)(1+y) = 1 + c_1 + c_2$

$c_1 = x + y$

$c_2 = xy$

denote $c_1(E) = a_1 \quad c_2(E) = a_2$

$c_1(E^*) = -a_1$

$\Rightarrow \quad 9 a_2 (2a_1^2 + a_2)$

use the identities

$$\begin{cases} 2 a_1 a_2 = a_1^3 & (I) \\ a_2^2 = a_1^2 a_2 & (II) \end{cases}$$

$\Rightarrow \quad 18 a_2^2 + 9 a_2^2 = 27 a_2^2$

$a_2^2 =$ class of a point on $Gr(2,4)$.

$a_2 = [$subspaces contained in $\mathrm{span}(e_1, e_2, e_3)]$

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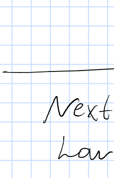
intersection is a single point. $\mathrm{span}(e_1, e_2)$.

Hence # of lines $= 27$.

$z_1 \quad z_2$



$$H^4(X, X \setminus z_1) \cdot H^4(X, X \setminus z_2)$$
$$\downarrow$$
$$H^8(X, X \setminus \{pt\})$$
$$\| $$
$$\mathbb{Z}$$

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# of lines on a quintic in $\mathbb{P}^4$ turns out to be 2875.

Next step: how many conics?

conic:

$V_0 s^2 + V_1 s t + V_2 t^2 \subset \mathbb{P}^4$

$(s,t) \in \mathbb{P}^1$

$V_0, V_1, V_2 \in \mathbb{C}^5$.

$Gr(3,5)$

$X(3,5) \to Gr(3,5)$

$\qquad\qquad\quad \Downarrow$

conics $\to V$

$\dim V = 3$

$\mathbb{P}(V) = \mathbb{P}^2$

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quintic restricted to V becomes a quintic curve on $\mathbb{P}^2$.

count cases when this curve splits as $2+3$

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conic through $0, 0, 1$

x^2, y^2, xy, xz, yz