Counting lines:

line in $\mathbb{P}^4$ is a point a $Gr(2,5)$

line is parametrized by $sV_1 + tV_2$ $\quad V_1, V_2 \in \mathbb{C}^5$

degree 5 equation $\to$ degree 5 polynomial in s, t.
So we have a section of the bundle F_5 whose fibers are degree 5 polynomials. If x, y are Chern roots of the dual tautological bundle in $Gr(2,5)$, then the Chern roots of F_5 are given by

$$5x,\ 4x+y,\ 3x+2y,\ 2x+3y,\ x+4y,\ 5y$$
$$\updownarrow$$
$$s^5 \quad s^4t \quad s^3t^2 \quad s^2t^3 \quad st^4 \quad t^5.$$

$$5x(4x+y)(3x+2y)(2x+3y)(x+4y)5y$$

express in terms of $x+y$, xy
$$x+y = c_1, \quad xy = c_2$$

modulo relations in $H^*(Gr(2,5))$ we divide it by the class of a point, obtain the number.

Counting rational quadratic curves on $Q^3 \subset$ degree 5 hypersurface in $\mathbb{P}^4$.

1) a quadratic curve is given by parametrization $s^2V_1 + stV_2 + t^2V_3$.
V_1, V_2, V_3 span a 3-dimensional subspace.
(otherwise we obtain just a parametrization of a line)

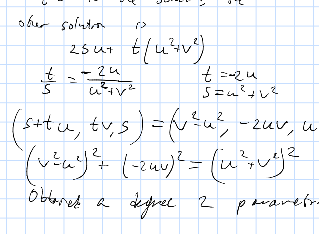
$\rightsquigarrow$ get a point on $Gr(3,5)$.
Going back, a point on $Gr(3,5)$ gives a $\mathbb{P}^2 \subset \mathbb{P}^4$,
so enumerating all quadratic curves in $\mathbb{P}^2$ gives all curves in $\mathbb{P}^4$,

$$\{\text{curves}\} \downarrow A, B, C, D, E, F \text{ coefficients of the equation.}$$
$$Gr(3,5)$$

Sometimes $Ax^2 + Bxy + \ldots = 0$ is a union of 2 lines or a single line.
Let's count these.

$\mathbb{P}^4$ — {curves} — fiber = all curves on $\mathbb{P}^2$. $Gr(3,5)$.
all degree 2 curves are rational!

$x^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 = 0 \subset \mathbb{P}^2$



For instance
$x^2 + y^2 = z^2$ contains $1, 0, 1$

$: s(1,0,1) + t(u,v,0)$
$= (s+tu, tv, s)$
$(s+tu)^2 + (tv)^2 = s^2$
$s^2 + 2stu + t^2u^2 + t^2v^2 = s^2$
$t=0$ is one solution, the other solution is
$2su + t(u^2+v^2)$
$\frac{t}{s} = \frac{-2u}{u^2+v^2}$ $\qquad t = -2u,\ s = u^2+v^2$
$(s+tu, tv, s) = (v^2-u^2, -2uv, u^2+v^2)$
$(v^2-u^2)^2 + (-2uv)^2 = (u^2+v^2)^2$
Obtained a degree 2 parametrization.

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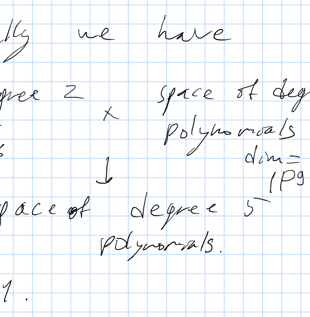
fix a point on $Gr(3,5)$
$\Downarrow$
restrict the equation of the quintic to the 3-dim subspace.

F_5 degree 5 polynomial in 3 variables.

we want to count when $F_5 = P \cdot Q$ $\quad P$ degree 2, Q degree 3.

is equivalent to counting conics.

$F_5 \to$



when does this curve of degree 5 splits as a union?

Geometrically we have

space of degree 2 polynomials ($\dim = 6$, $\mathbb{P}^5$) $\times$ space of degree 3 polynomials ($\dim = 10$, $\mathbb{P}^9$)
$\downarrow$
space of degree 5 polynomials.

$$\binom{7}{2} = 21.$$

$$\mathbb{P}^5 \times \mathbb{P}^9 \longrightarrow \mathbb{P}^{20}$$

vary as a point on $Gr(3,5)$ varies.

$\mathbb{P}^5$ – bundle on $Gr(3,5)$
$\mathbb{P}^9$ – bundle on $Gr(3,5)$.

$\mathbb{P}^5 \times \mathbb{P}^9$ bundle
$$X_{5,9} \to Gr(3,5)$$
$$\searrow X_{20} \to Gr(3,5)$$

$S: Gr(3,5) \to X_{20}$ section

we need to intersect $Im(S)$, $\dim Im(S) = 6$
with $X_{5,9}$

$\dim X_{5,9} = 6 + 5 + 9 = 20$
inside X_{20} $\quad \dim = 26$.

Strategy:
Compute the class of $Im(S)$, restrict it to $X_{5,9}$, divide by the class of a point in $H^{40}(X_{5,9})$.

$$X_{5,9} \longrightarrow X_{20} \supset Im(S).$$
$$\mathbb{P}^5 \times \mathbb{P}^9 \downarrow \quad \swarrow \mathbb{P}^{20}$$
$$Gr(3,5)$$

<u>Question</u> $V \to X$ vector bundle
S a section which is never $0 \rightsquigarrow$ produces a map $X \to \mathbb{P}(V)$, partial inverse of $\mathbb{P}(V) \to X$.
what is the cohomology class of $S(X)$? $\quad rk\, V = r$

$$\pi: \mathbb{P}(V) \to X.$$

$$0 \to \tau_{\mathbb{P}(V)} \to \pi^* V \to V' \to 0$$
V' is a quotient.
S induces a section of V', which vanishes precisely over the $S(X)$.

If V has chern classes $1 + c_1(V) + \ldots$, then V' has Chern classes

$$\frac{1 + c_1(V) + c_2(V) + \ldots}{1 + c_1(\tau_{\mathbb{P}(V)})}$$

$$[S(X)] = \left(\frac{1 + c_1(V) + \ldots}{1 + c_1(\tau_{\mathbb{P}(V)})}\right)_{\text{degree } r-1 \text{ term.}}$$

$$\frac{1}{1 + c_1(\tau_{\mathbb{P}(V)})} = 1 - c_1(\tau_{\mathbb{P}(V)}) + c_1(\tau)^2 \ldots$$

Obtain
$$[S(X)] = c_{r-1}(V) - c_{r-2}(V)c_1(\tau) + c_{r-3}(V)c_1(\tau)^2 + \ldots \quad (*)$$

So we know the class of $S(Gr(3,5))$, we need to pullback to $X_{5,9}$.

Since V comes from $Gr(3,5)$, its Chern classes are in
$$H^*(Gr(3,5)) \subset H^*(X_{20})$$
$$\downarrow$$
$$\subset H^*(X_{5,9}).$$

what happens with $c_1(\tau)$
$$X_{5,9} \xrightarrow{f} X_{20}$$

$$f^* c_1(\tau) = c_1(f^*\tau).$$
$$f^* \tau_{X_{20}} = \tau^{(1)}_{X_{5,9}} \otimes \tau^{(2)}_{X_{5,9}}$$

$X_{5,9} \xrightarrow{\mathbb{P}^9} X_5$, $X_{5,9} \xrightarrow{\mathbb{P}^5} X_9$, $X_5 \xrightarrow{\mathbb{P}^5} Gr(3,5)$, $X_9 \xrightarrow{\mathbb{P}^9} Gr(3,5)$

$\tau^{(1)}_{X_{5,9}}$ = pullback of τ_{X_5}
$\tau^{(2)}_{X_{5,9}}$ = pullback of τ_{X_9}.

$$F_5(x,y,z) = F_2(x,y,z)\, F_3(x,y,z).$$

$$c_1(f^*\tau_{X_{20}}) = c_1(\tau^{(1)}_{X_{5,9}}) + c_1(\tau^{(2)}_{X_{5,9}}). \quad (*)$$

So we are reduced to a computation in $H^*(X_{5,9})$.

$$H^*(X_{5,9}) = H^*(Gr(3,5))[c_1(\tau^{(1)}_{X_{5,9}}), c_1(\tau^{(2)}_{X_{5,9}})]$$
$$c_1(\tau^{(1)}_{X_{5,9}}) = u, \quad c_1(\tau^{(2)}_{X_{5,9}}) = v$$

$H^*(Gr(3,5))$: polynomials in 3 variables modulo equations

x, y, z satisfying
$$(1+x)(1+y)(1+z) = \sum c_i(E) = 1 + a_1 + a_2 + a_3$$
$$(1 + a_1 + a_2 + a_3)(1 + b_1 + b_2) = 1$$

$$H^*(X_5) = H(Gr(3,5)) \oplus uH(Gr(3,5)) \oplus \ldots \oplus u^4 H(Gr(3,5))$$
how does u^5 express in terms of $u, u^2, \ldots$?

for a projective bundle:
Since $(1+u)(1 + c_1(V') \ldots) = \sum c_i(V)$, with $u = c_1(\tau)$ and $(1 + c_1(V') \ldots)$ a polynomial of degree 1 less
$$u^5 - u^4 c_1(V) + \ldots = 0$$