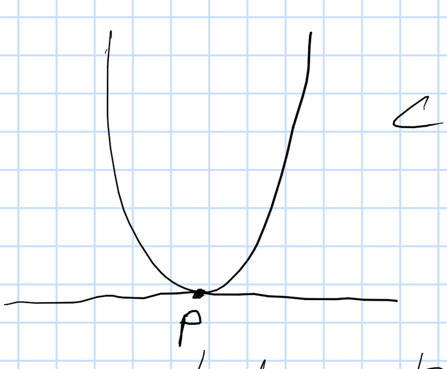
Z_1 $y=x^2$

Z_2 $y=0$



$\subset \mathbb{C}^2$

$Z_1 \cap Z_2$ what is the class of the intersection ?

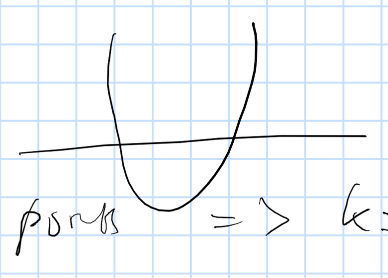
$H^4(\mathbb{C}^2, \mathbb{C}^2 \setminus p; \mathbb{Z})$ the class must be a k. the generator of $H^4(\mathbb{C}, \mathbb{C}^2 \setminus p; \mathbb{Z})$. What is k ?

look at $H^4(\mathbb{C}P^2)$.

$y=x^2$ becomes $yz=x^2$ ← intersecting a quadratic curve

$y=0$ becomes $y=0$ ← line

in $H^4(\mathbb{C}P^2)$ doesn't change if we replace these curves by different curves



this has 2 simple intersection points $\Rightarrow k=2$.

When we count some things by computing in cohomology, for instance $S \subset X$ S a set of points $|S| \cdot * = [S] \in H^{2d}(X, \mathbb{Z}) \Rightarrow$ works only $H^{2d}(X,\mathbb{Z}) \neq \{0\} \Rightarrow X$ must be compact.

removing a point from a compact manifold kills H^{2d}

$$H^{2d}(X, X \setminus pt) \to H^{2d}(X) \to H^{2d}(X \setminus pt) \to 0$$
$$\mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \qquad\qquad \overset{\|}{0}$$

So for enumerative geometry we need compact spaces.

<u>Today</u> we consider conics (including degenerate)

this compactification of the space of conics is not the best to generalize. Kontsevich moduli space is better. (for the next time).

Conics are given by projective equations

$$P = Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0$$

$$M = \begin{pmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{pmatrix} \rightsquigarrow (x\ y\ z)\ M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

$\det M =$ called the discriminant of the equation.

$\det M \neq 0 \iff$ equation is irreducible $\Rightarrow$ set of solutions is a smooth manifold

$\det M = 0 \Rightarrow \operatorname{rank} M = 0, 1, 2$

Assume $(A,B,C,D,E,F) \neq (0,0,0,0,0,0) \Rightarrow$ remains

2 cases $\operatorname{rank} M = 1$ or $\operatorname{rank} M = 2$.

$P = L_1^2$ $P = L_1 L_2$ $L_1 \neq \mathbb{C} \cdot L_2$

L_1 linear. L_1, L_2 linear

Conic curves in $\mathbb{R}^2$.

On $\mathbb{R}P^2$ we have a large group of transformations ⇝ 3×3 real matrices. (8 dimensional group).

circles, parabolas, ellipses, hyperbolas ?

euclidean motions + scaling is $2+1+1 = 4$ dimensional.

1) euclidean + scaling correspond to transformations preserving every point of the line at ∞.

geometrically line at ∞ corresponds to "directions"

goes to ∞

2) rotations correspond to orthogonal matrices which preserve the line at ∞.

points at ∞ look like $(x, y, 0)$

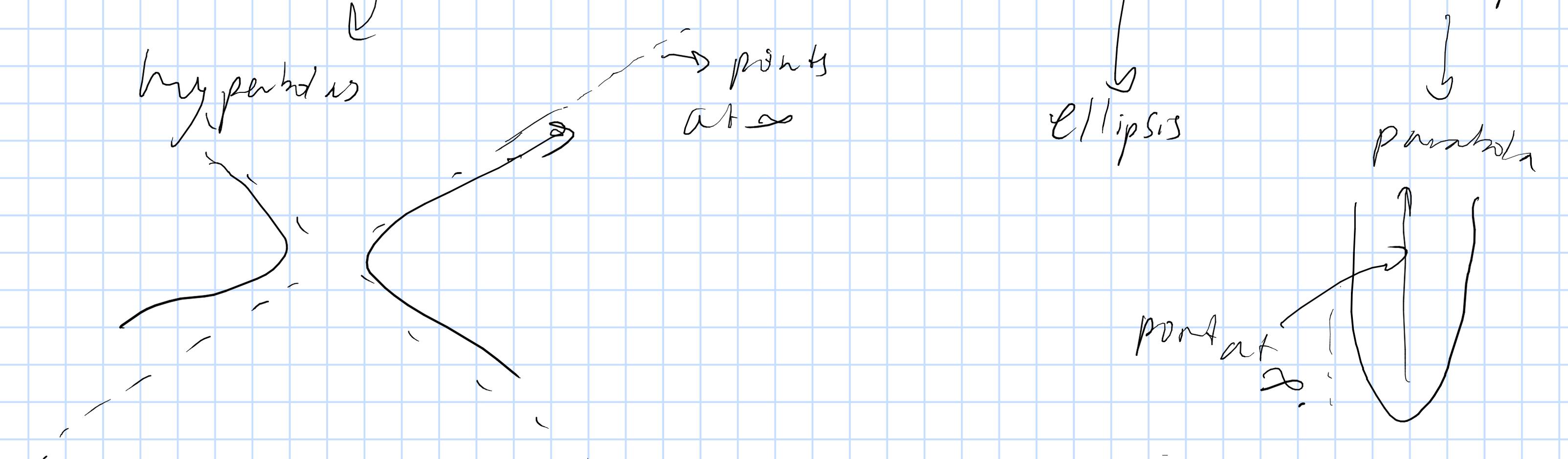
$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

a <u>circle</u> is a conic passing through the points $(1, i, 0)$ and $(1, -i, 0)$.

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos\varphi + i\sin\varphi \\ -\sin\varphi + i\cos\varphi \end{pmatrix} = (\cos\varphi + i\sin\varphi) \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Any real conic intersects the line at ∞ at either 2 real points or 2 complex conjugate points or 1 double point.

hyperbolas   ellipsis   parabola



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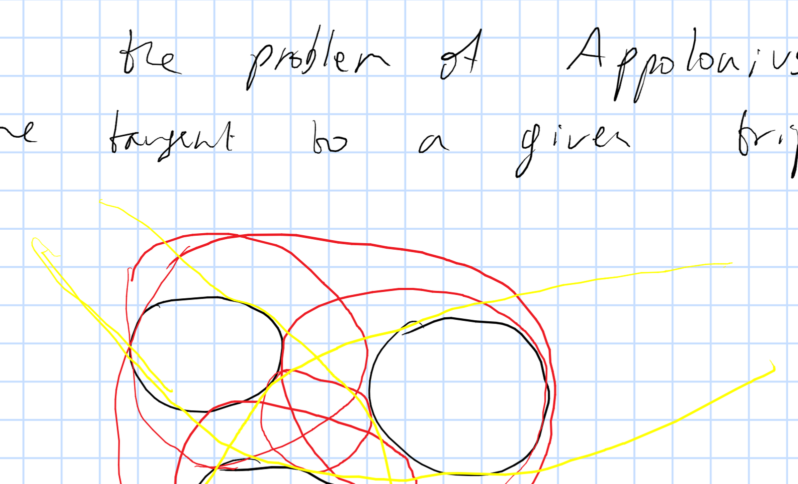
the problem of Appolonius : How many circles are tangent to a given triple of circles?



$1+1+3+3=8$.

circles are conics passing through $(1, i, 0)$ $(1, -i, 0)$.

all conics are parametrized by points of $\mathbb{P}^5$. the condition that conic passes through some point is linear

$$Ax_0^2 + Bx_0y_0 \ \ldots = 0 \quad \text{linear in } A,B,C,D,E,F.$$

$\Rightarrow$ circles are parametrized by points of $\mathbb{P}^3$.

How to test that our conic is tangent to a given other conic ?

Parametrize the other conic $\mathbb{P}^1 \xrightarrow[\text{degree 2 map}]{} \mathbb{P}^2$

$$(\alpha_{11}s^2 + \alpha_{12}st + \alpha_{13}t^2 : \alpha_{21}s^2 + \ldots : \quad)$$

Example: $(x^2-y^2 : 2xy : x^2+y^2)$

substitute this in $P(x,y,z) = Ax^2 + Bxy + \ldots$

obtain degree 4 polynomial in s, t.

$$f_4 s^4 + f_3 s^3 t + f_2 s^2 t^2 + f_1 s t^3 + f_0 t^4 = F$$

f_i are linear polynomials in $A, B, \ldots F$.

generally this polynomial will factor in $\prod_{i=1}^4 (a_i s + b_i t)$.

Notice that $(1, i, 0)$ $(i, 1, 0)$ are always 2 intersection points, we can divide F by the corresponding linear polynomials obtain degree 2 polynomial in s, t

$f_2 s^2 + f_1 st + f_0 t^2$. solutions are the intersection points in $\mathbb{C}^2 \subset \mathbb{P}^2$.

tangency condition $\Rightarrow$ $f_1^2 = 4 f_0 f_2$.

given by degree 2 polynomial in $A, B, \ldots, F$.

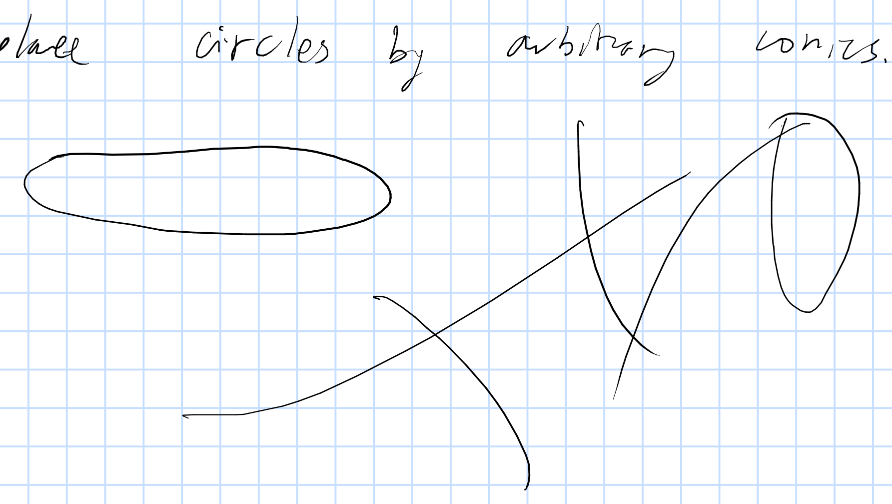
$=$ degree 2 surface in $\mathbb{P}^3$.

intersection of 3 degree 2 surfaces has 8 points (unless something goes wrong).

Break 15:56.

$3264 = ?$

replace circles by arbitrary conics.



How many conics? $\mathbb{P}^5$

need 5 fixed conics $C_1, C_2, \dots C_5$.

One conic varies. How many are tangent to all 5?

conic $\to$ parametrized by $\mathbb{P}^5$. (A,B,C,D,E,F)

a fixed conic is given by parametrization $(\alpha_{11}s^2+\alpha_{12}st+\alpha_{13}t^2 : \dots)$

intersection is given by

$\Rightarrow$ degree 4 polynomial $f_4 s^4 + f_3 s^3 t + \dots + f_0 t^4$ $(*)$

f_i are linear in $A,B,C,\dots F$.

When does $(*)$ have double roots?

can replace $f_4 s^4 + \dots$ by $f_4 x^4 + f_3 x^3 + f_2 x^2 + f_1 x + f_0 = 0$

$\mathbb{P}^1(s,t) \to \mathbb{C}^1\ (x)$

$s,t \mapsto x = \frac{s}{t}$

$(x,1) \leftarrow x$

[∞ is a tangent point if $f_4 = f_3 = 0$]

Recall the theory of discriminants.

Consider a monic equation

$$x^n + a_{n-1}x^{n-1} + \dots + a_0 = \prod_{i=1}^{n}(x-x_i)$$

Consider $D(x_1,\dots,x_n) = \prod_{i\neq j}(x_i - x_j)$. Permuting $x_1,\dots,x_n$ doesn't change D.

Ex.

$n=2$: $\boxed{D(x_1,x_2) = (x_1-x_2)(x_2-x_1) = -x_1^2+2x_1x_2-x_2^2}$

Hence it is possible to express D in terms of $a_{n-1},\dots,a_0$.

$n=2$: $-(x_1+x_2)^2+4x_1x_2 = -a^2+4b$ $\quad(-(a^2-4b))$

x^2+ax+b

usual definition $D = \prod_{i<j}(x_i-x_j)^2$.

$D = D(a_0,\dots,a_{n-1})$. $\quad$ ($D=0 \Leftrightarrow$ polynomial has double roots)

but we have not a monic polynomial

$f_n x^n + \dots + f_0$ has double roots $\Leftrightarrow D\left(\frac{f_0}{f_n},\dots,\frac{f_{n-1}}{f_n}\right) = 0$

has some denominator $f_n^{?}$. need to multiply by some power of f_n. which power?

Solution: decompose $f_n x^n + \dots + f_0$ as $\prod(a_i x - b_i)$,

write $\tilde{D} = \prod_{i\neq j}(a_i b_j - a_j b_i)$. Claim: is a polynomial of $f_0,\dots,f_n$.

Proof: $\tilde{D} = \prod a_i a_j\left(\frac{b_j}{a_j} - \frac{b_i}{a_i}\right) = \underbrace{f_n^{2(n-1)} \cdot D\left(\frac{f_0}{f_n},\dots,\frac{f_{n-1}}{f_n}\right)}_{(*)}$

can only have f_n in the denominator.

switching the roles of a's and b's we see that $\tilde{D}$ can only have f_0 in the denominator.

$\Rightarrow \tilde{D}$ doesn't have any denominator. $\square$

! $(*) \Rightarrow \tilde{D}$ is homogeneous of degree $2(n-1)$.

$\tilde{D} = 0 \Rightarrow a_i b_j = a_j b_i$ some i,j $\Leftrightarrow f_n s^n + \dots + f_0 t^n$ has a double root.

for our case $n=4$ $\quad$ degree of $\tilde{D} = 6$.

So the condition that a conic touches another conic has degree 6.

Intersection of 5 degree 6 equations should give $6^5 = 7776$ points ($\neq 3264$!).

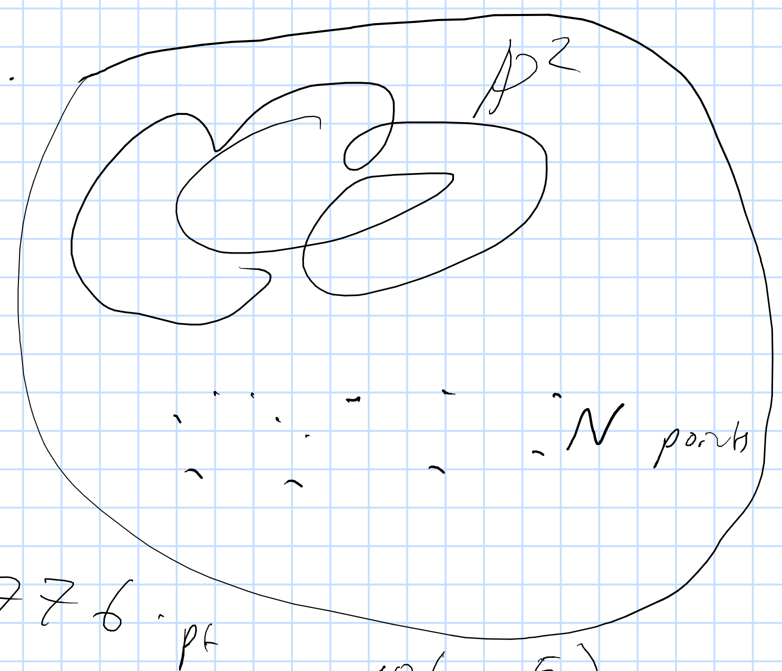
So things do go wrong here!

Suppose A,B,C,D,E,F defines a double line, $(\mathrm{rk}\, M = 1) \Rightarrow$ of course we have double roots.

$A(\dots) + B \dots = (\dots)^2$

given by $(\alpha x + \beta y + \gamma z)^2$ $\quad (\alpha,\beta,\gamma) \in \mathbb{P}^2$.

$\mathbb{P}^2 \subset \mathbb{P}^5$ "false positives".



2 ways out

1) compute how much does $\mathbb{P}^2$ contribute?

embed nbh of $\mathbb{P}^2$ somewhere else, compute the contribution.

$\to$ excess intersection.

Answer: $N = 7776 -$ excess

$6^5 = 7776$. pt $\in H^{10}(\mathbb{P}^5)$.

2) work with a different compactification (instead of $\mathbb{P}^5$)

Should we do SAGE next time?