1 part of SAGE

we found that

for fixed conic C

all lines tangent to C form a conic C'

lines in $\mathbb{C}^3$ $= Gr(2,3)$
$=$ 1-dimensional subspaces in the dual space.

Notation;

V original space

$\mathbb{P}(V)$ original proj space

$\mathbb{P}(V^*)$ dual projective space.

point on $\mathbb{P}(V) \longleftrightarrow$ lines in $\mathbb{P}(V^*)$

lines $\qquad$ points.

C conic on $\mathbb{P}(V) \Rightarrow C^*$ dual conic on $\mathbb{P}(V^*)$.

Consider the closure in $\mathbb{P}^5 \times \mathbb{P}^5$ of the set of pairs (C, C^*). As a set it contains the following pairs:

1) C, C^* $\quad$ C non-degenerate conic
C^* the dual

2) $C = \ell^2$, pair of lines whose intersection point corresponds to ℓ.

3) opposite of 2)

4) $C = \ell^2$, ℓ'^2 such that ℓ' corresponds to a point on ℓ.

We can replace $\mathbb{P}^5$ by this new space, hope that the condition to be tangent to 5 conics doesn't have extra solutions there.

Blow-ups

General idea $\quad Z \subset X \quad X \setminus Z = U$

what we care about

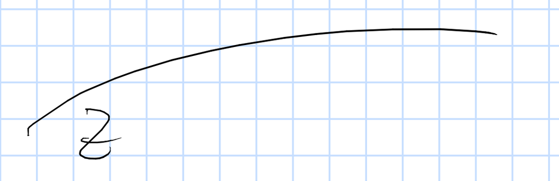
Z is a "bad locus".

contains excess intersection.

we added Z to get a compact space.

We blow up Z, replacing it by something bigger, hope to get better intersection.

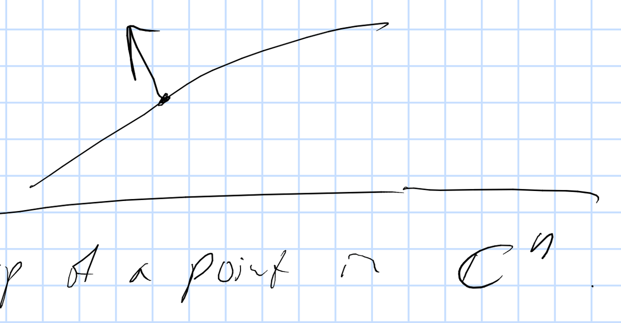
Picture $\quad X$



points of $Bl_Z X$ are:

1) points of $X \setminus Z$
2) pairs (point z of Z, line in the normal vector space to $z \in Z$).

$T_z Z \subset T_z X \qquad N_{z,Z} X = T_z X / T_z Z$



Blow up of a point in $\mathbb{C}^n$.

0

Take pairs (line through 0, point on that line)

denote this space by $Bl_0 \mathbb{C}^n$.

$Bl_0 \mathbb{C}^n$ is isomorphic to the tautological bundle on $\mathbb{CP}^{n-1}$.

Consider the map $\pi : Bl_0 \mathbb{C}^n \to \mathbb{C}^n$ (forget the line).

outside of 0 $\quad \pi^{-1}(\mathbb{C}^n \setminus \{0\}) \cong \mathbb{C}^n \setminus \{0\}$

over 0 we have $\pi^{-1}(\{0\}) = \mathbb{CP}^{n-1}$.

In general $Z \subset X$ $\quad Z$ smooth

$z_0 \in Z$ we can choose a neighborhood so that $Z \subset X$ looks like $\mathbb{C}^k \subset \mathbb{C}^n$ as a coordinate subspace

$\mathbb{C}^k = (e_1, \dots, e_k)$

$\mathbb{C}^n = (e_1, \dots, e_n)$.

$Bl_{\mathbb{C}^k} \mathbb{C}^n = \mathbb{C}^k \times Bl_0 \mathbb{C}^{n-k}$.

Tricky question: to glue these together using <u>holomorphic</u> maps (polynomial maps).

In general we obtain $\pi : Bl_Z X \to X$ so that $\pi|_{\pi^{-1}(X \setminus Z)}$ is isomorphism.

$\pi^{-1}(Z) \cong P(N_{Z \subset X})$

normal bundle

$T_Z \subset T_X|_Z \qquad N_{Z \subset X} = T_X / T_Z$.

In enumerative geometry we "replace difficult spaces" by their cohomology rings.

So what do some operations on spaces do to the cohomology?

Before we have seen: $\mathcal{E} \to X$

$$H^*(P(\mathcal{E})) = H^*[X][\underset{\text{dual taut. bundle}}{c_1(\mathcal{O}(1))}] \big/ \eta^n + c_1(\mathcal{E})\eta + \dots + c_n(\mathcal{E})$$

with $\eta = c_1(\mathcal{O}(1))$.

$$H^*[X] \oplus \eta H^*[X] \oplus \dots \eta^{n-1} H^*[X].$$

$\eta \qquad \eta$

What about the Blow-up?

$$\begin{array}{ccc} P(N_{Z \subset X}) & \subset & \tilde{X} := Bl_Z X \\ \downarrow & & \downarrow \\ Z & \subset & X \end{array}$$

$\exists \quad U \supset Z$ open such that $U \underset{\text{homeomorphic}}{\simeq} N_{Z \subset X}$

How does $\pi^{-1}(U)$ look like?

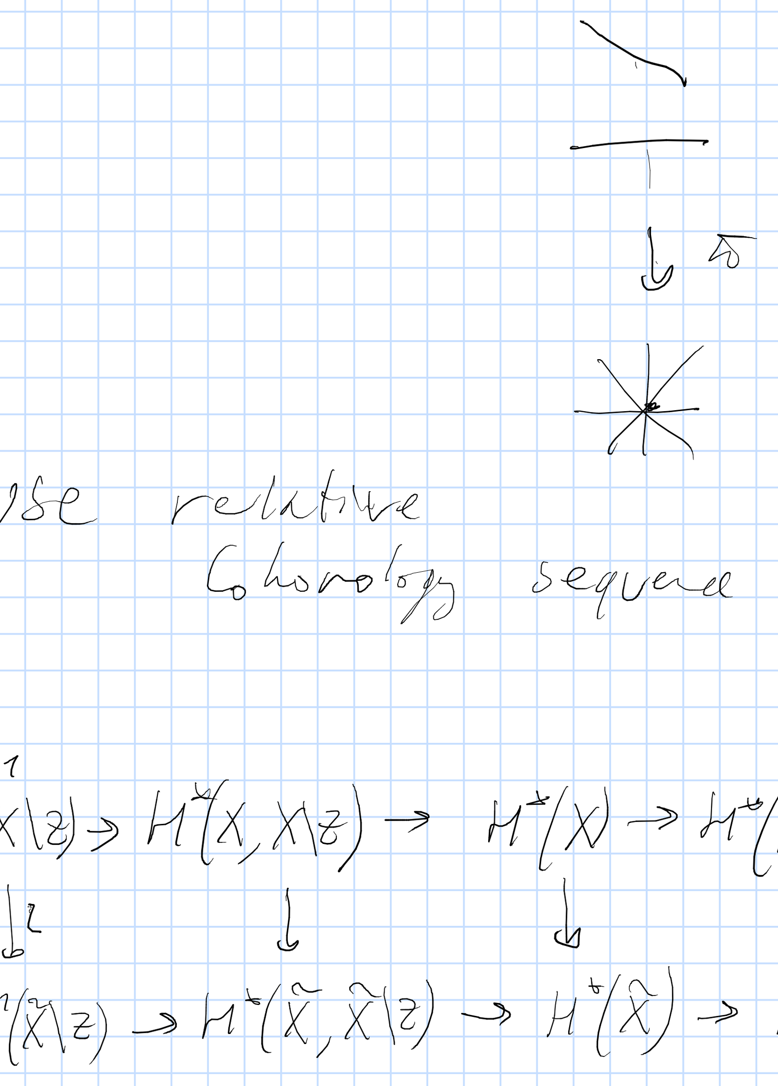
$\pi^{-1}(U) = Bl_Z U \cong Taut(P(N_{Z \subset X}))$

$P(N_{Z \subset X}) \subseteq Taut(P(N_{Z \subset X})) \cong Bl_Z U$

line bundle

$Z \subseteq N_{Z \subset X} \cong U$

π_N



Use relative Cohomology sequence for $X, X \setminus Z$

$\tilde{X}, \tilde{X} \setminus Z$

$$\begin{array}{ccccccc} H^{*-1}(X \setminus Z) & \to & H^*(X, X \setminus Z) & \to & H^*(X) & \to & H^*(X \setminus Z) \\ \downarrow \wr & & \downarrow & & \downarrow & & \downarrow \wr \\ H^{*-1}(\tilde{X} \setminus Z) & \to & H^*(\tilde{X}, \tilde{X} \setminus Z) & \to & H^*(\tilde{X}) & \to & H^*(\tilde{X} \setminus Z) \end{array}$$

Let's understand

$H^*(X, X \setminus Z) \longrightarrow H^*(\tilde{X}, \tilde{X} \setminus Z)$

by excision replace X by U

$\tilde{X}$ by $Bl_Z U = \tilde{U}$.

$$\begin{array}{ccc} H^*(N_{Z \subset X}, N_{Z \subset X} \setminus Z) & \to & H^*(Taut\, P(N), Taut\, P(N) \setminus P(N)) \\ \wr \uparrow w_U & & \wr \uparrow w' \\ H^{*-2rkN}(Z) & & H^{*-2}(P(N)) \\ & & \wr \\ & & H^{*-2}(Z) \oplus H^{*-4}(Z)\eta \oplus \dots \\ & & H^{*-2n}(Z)\eta^{n-1} \end{array}$$

$n = rkN$

$$\begin{array}{ccc} H^*(N_{Z \subset X}, N_{Z \subset X} \setminus Z) & \to & H^*(Taut\, P(N), \dots) \\ \uparrow w & & \uparrow \\ H^{*-2n}(Z) & \xrightarrow{\eta^{n-1}} & H^{*-2}(P(N)) \end{array}$$

commutes.