Blow up $\mathbb{P}_Z \subset \tilde{X} \supset U$

$Z \subset X \supset U$

$$\begin{tikzcd}
 & H^*(Z) \arrow[d, phantom, "\cong"] & & \\
H^{i-1}(U) \arrow[r] \arrow[d, "\wr"] & H^i(X,U) \arrow[r] \arrow[d] & H^i(X) \arrow[r] \arrow[d] & H^i(U) \arrow[d, "\wr"] \\
H^{i-1}(U) \arrow[r] & H^i(\tilde{X},U) \arrow[r] \arrow[d, phantom, "\cong"] & H^i(\tilde{X}) \arrow[r] & H^i(U) \\
 & H^i(\mathbb{P}_Z) & &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\wr"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "1"] & D \arrow[r] \arrow[d, "\wr"] & \tilde{B} \arrow[d] \\
A \arrow[r] & B\oplus B' \arrow[r, "2"'] & E \arrow[r] & D \arrow[r] & \widetilde{B\oplus B'}
\end{tikzcd}$$

1 is injective: clear

2 restricted to B' is injective.

$B' \cap C$ in E is zero.

$B' + C = E$:

Proved: $E \cong B' \oplus C$

Corollary: $H^i(Bl_Z X) \cong H^i(X) \oplus \underbrace{H^{i-2}(Z) \oplus H^{i-4}(Z) \dots}_{\text{codim } Z - 1 \text{ parts.}}$

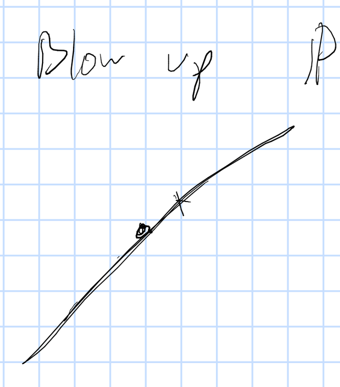
Recall we had

$\mathbb{P}^5$ parametrizes conics in $\mathbb{P}^2$

$\mathbb{P}^2 \subset \mathbb{P}^5$ double lines

$(ax+by+cz)^2 = 0$

Blow up $\mathbb{P}^2$:



$\mathbb{P}^5$

$(ax+by+cz)^2 + \varepsilon(Ax^2 + Bxy + \dots) = 0$

Strategy: understand how discriminant behaves as $\varepsilon \to 0$

$D(\varepsilon)$ goes to 0 $\Rightarrow$

$\frac{D(\varepsilon)}{\varepsilon^2}$ hope that this is not 0.

give a section of

$F \otimes \mathcal{L}$

original vector bundle

line bundle whose sections have behavior $\sim \frac{1}{\varepsilon^2}$ along $\pi^{-1}(Z) \subset \hat{X}$.

D has zero along $\mathbb{P}_Z$ in $\hat{X}$ of order 2

Suppose $\mathcal{L}$ line bundle which has a section s which has a pole of order 2 along $\mathbb{P}_Z$

$\mathcal{L}$ is dual to the line bundle with section which has a zero of order 2 along $\mathbb{P}_Z$

$\Rightarrow c(\mathcal{L}^*) = 1 + 2[\mathbb{P}_Z]$

$c(D) = 1 + 6[h]$ h class of hyperplane on $\mathbb{P}^5$

$c(D\mathcal{L}) = 1 + 6[h] - 2[\mathbb{P}_Z]$

$$\begin{tikzcd}
\tilde{X} \arrow[d] \\
\mathbb{P}^5 \ni h
\end{tikzcd}$$

$(6[h] - 2[\mathbb{P}_Z])^5$

$H^*(\tilde{X}) = \underset{\mathbb{Q}[h]/h^6=0}{\underset{\shortparallel}{H^*(\mathbb{P}^5)}} \oplus H^*(\mathbb{P}^2) \oplus H^*(\mathbb{P}^2)$

$H^*(\mathbb{P}^2)$ is generated by $\tilde{h}$

$H^2(\tilde{X}) = H^2(\mathbb{P}^5) \oplus H^0(\mathbb{P}^2) \ni [\mathbb{P}_Z]$

$H^4(\tilde{X}) = H^4(\mathbb{P}^5) \oplus H^2(\mathbb{P}^2) \oplus H^0(\mathbb{P}^2)$

$H^i(X) \xrightarrow{\text{pullback}} H^i(\tilde{X})$

$6^5[pt]$ $[h]$

natural map.

$H^i(\tilde{X}, U)$ $[\mathbb{P}_Z]$

$[\mathbb{P}_Z] \cdot [h]$ $\mathbb{P}_Z \to Z \subset X$

is $2\tilde{h}$ $[h]$ restricted to $Z \cong \mathbb{P}^2$

$\underset{Z}{\underset{\shortparallel}{\mathbb{P}^2}} \subset \mathbb{P}^5$

$a, b, c \to$ coeffs of $(ax+by+cz)^2$

$H^*(\mathbb{P}_Z) = H^*(Z)[\eta]$ $\eta = c_1(\text{taut bundle})$

$\eta^3 - c_1\eta^2 + c_2\eta = 0$ $(c_3 = 0)$

chern classes of the normal bundle to $\mathbb{P}^2$ in $\mathbb{P}^5$.

normal bundle = tangent bundle to $\mathbb{P}^5$ − tangent bundle to $\mathbb{P}^2$.

$\frac{(1+h)^6}{(1+\tilde{h})^3} = \frac{(1+2\tilde{h})^6}{(1+\tilde{h})^3}$ tangent bundle to $\mathbb{P}^n$: $0 \to O(1)^{\oplus(n+1)}$

$= 1 + 9\tilde{h} + 30\tilde{h}^2$

$H^*(\mathbb{P}_Z) = \mathbb{Q}[\eta, \tilde{h}] / \tilde{h}^3 = 0,\ \eta^3 - 9\eta^2\tilde{h} + 30\eta\tilde{h}^2 = 0$

generator of $H^*(X, X \setminus \mathbb{P}_Z) \simeq H^i(\mathbb{P}_Z)$

$[\mathbb{P}_Z] \rightsquigarrow 1 \in H^0(\mathbb{P}_Z)$

multiplying by $h \to$ multiplying by $2\tilde{h}$

multiplying by $[\mathbb{P}_Z] \to$ multiplying by η.

$H^*(X, X \setminus \mathbb{P}_Z) \simeq H^*($Tot Taut

$(6h - 2[\mathbb{P}_Z])^5 = 2^5(3h - [\mathbb{P}_Z])^5$

$= 2^5\big(3^5 - \cancel{5 \cdot 3^4 \cdot (h^4[\mathbb{P}_Z])} + \cancel{10 \cdot 3^3(h^3\eta[\mathbb{P}_Z])}$

$- 10 \cdot 3^2(h^2\eta^2[\mathbb{P}_Z])$

$+ 5 \cdot 3(h\eta^3[\mathbb{P}_Z])$

$- \eta^4[\mathbb{P}_Z]$

$h\eta^3 = 2\tilde{h} \cdot 9\eta^2\tilde{h} = 18\eta^2\tilde{h}^2$

$h^2\eta^2 = 4\tilde{h}^2 \cdot \eta^2$

$\eta^4 = \eta(9\eta^2\tilde{h} - 30\eta\tilde{h}^2)$

$= -30\eta^2\tilde{h}^2 + 81\eta^2\tilde{h}^2 = 51\eta^2\tilde{h}^2$

$3^5 - 90 \cdot 4 + 15 \cdot 18 - 51 = 102$

$2^5 \cdot 102 = 3264$!

= # of conics tangent to 5 conics.

$Bl_{\mathbb{P}^2}(\mathbb{P}^5)$ = Space of (conic, dual conic) from before.