$$\left(\frac{x}{1-e^{-x}}\right)^{n+1} e^{mx} \underset{\text{take the coeff of } x^n}{} = \binom{n+m}{m} \quad (*).$$

Proof of Grothendieck-Riemann-Roch in general case will depend on the result for line bundles on $\mathbb{CP}^n$.

Main tool residue calculations.

Consider $\mathbb{C}((z))$ = the field of formal Laurent series. $f \in \mathbb{C}((z))$ looks like

$f = c_k z^k + c_{k+1} z^{k+1} + \ldots$ k can be negative.

field operations: $+, -, \cdot$ clear

$f \neq 0$ assume $c_k \neq 0$

$$f = c_k z^k \left(1 + \frac{c_{k+1}}{c_k} z + \ldots\right) \qquad g := \frac{c_{k+1}}{c_k} z + \ldots$$

$$f^{-1} = \frac{1}{c_k z^k}\left(1 - g + g^2 - g^3 \ldots\right).$$

Consider differential forms of the type $f(z)\,dz$.
(space of diff. forms $= \mathbb{C}((z))$ by definition).

Residues: $\operatorname{Res} f(z)\,dz :=$ coefficient of z^{-1} in f.

differential: $d: \mathbb{C}((z)) \to \mathbb{C}((z))\,dz$ is the operation sending $f(z)$ to $f'(z)\,dz$.

change of variable. Let $h(z) \in z\underbrace{\mathbb{C}[[z]]}_{\text{power series (no negative degrees of } z)}$, i.e.

h is a series with 0 constant term. no negative degrees.

for any $f \in \mathbb{C}((z))$ we can substitute $f(h(z))$

If $f = f_k \cdot z^k + \ldots$ $\quad f(h(z)) = f_k h(z)^k + f_{k+1} h(z)^{k+1} + \ldots$

when i is large $f_i h(z)^i$ has only terms of degree $\geq i$.

So to compute any coefficient of $f(h(z))$, we need to sum finitely many numbers.

A very powerful machinery to prove identities.

Main Lemma

$h(z)$ as before, $f(h(z))\,dh(z) := f(h(z))h'(z)\,dz$.

Then $\operatorname{Res} f(z)\,dz = \operatorname{Res} f(h(z))\,dh(z) \cdot \frac{1}{\operatorname{ord} h}$, where $\operatorname{ord} h =$ minimal z-degree in h.

Proof. • $\operatorname{Res} f(z)\,dz = 0 \iff f(z)\,dz = d(\tilde{f}(z))$ for some $\tilde{f}(z) \in \mathbb{C}((z))$.

• $\operatorname{Res} f(z)\,dz = 0 \Rightarrow$

$(k<0)$ $\quad f(z) = f_k z^k + \ldots + f_{-2} z^{-2} + f_0 + f_1 z + \ldots$

Let $\tilde{f}(z) = \frac{f_k}{k+1} z^{k+1} + \ldots + \frac{f_{-2}}{-1} \cdot z^{-1} + f_0 z + \frac{f_1 z^2}{2} + \ldots$

$d\tilde{f}(z) = f(z)\,dz$ clearly.

$\Leftarrow$ $f(z)\,dz = d(\tilde{f}(z))$, $\tilde{f}(z) = \sum \tilde{f}_k z^k$

then $f(z) = \sum \tilde{f}_k \cdot k \cdot z^{k-1}$. Clearly $\operatorname{Res} f(z)\,dz = 0$

• any f: $\quad f(z)\,dz = (\operatorname{Res} f(z)\,dz)\frac{dz}{z} + d\tilde{f}(z)$.

we have $\quad f(h(z))\,dh(z) = (\operatorname{Res} f(z)\,dz)\dfrac{dh(z)}{h(z)} + d\tilde{f}(h(z)) \Rightarrow$

using
$$\begin{array}{ccc} \mathbb{C}((z)) & \xrightarrow{\text{subst. } h} & \mathbb{C}((z)) \\ \downarrow d & & \downarrow d \\ \mathbb{C}((z))dz & \xrightarrow{\text{subst. } h} & \mathbb{C}((z))dz \end{array}$$
commutes. (because
$$\begin{array}{ccc} z^k & \to & h(z)^k \\ \downarrow & & \downarrow \\ k \cdot z^{k-1}dz & \to & (h(z)^k)'dz \end{array}$$
$(h(z)^k)' = k \cdot h(z)^{k-1} \cdot h'(z)$. So commutes)

hence
$$\operatorname{Res} f(h(z))\,dh(z) = (\operatorname{Res} f(z)\,dz) \cdot \operatorname{Res} \frac{dh(z)}{h(z)}$$

$$\operatorname{Res} \frac{dh(z)}{h(z)} = \operatorname{Res} \frac{h_m \cdot m \cdot z^{m-1} + \ldots}{h_m \cdot z^m + \ldots}\,dz = m.$$

$h(z) = h_m z^m + \ldots$

$h_m \neq 0 \quad m \geq 1$

So $\operatorname{Res} f(h(z))\,dh(z) = (\operatorname{Res} f(z)\,dz) \cdot \underset{m}{\underset{\shortparallel}{\operatorname{ord} h}}$.

The case $m = \operatorname{ord} h = 1$ is special because

for $h(z) = h_1 z + h_2 z^2 + \ldots$ $\quad h_1 \neq 0$

$\exists!\ h^*(z)$ s.t. $\quad h(h^*(z)) = h^*(h(z)) = z$.

$$\left(\frac{x}{1-e^{-x}}\right)^{n+1} e^{mx} \underset{\text{take the coeff of } x^n}{} = \binom{n+m}{m}. \quad (*).$$

$$LHS = \operatorname{Res} \left(\frac{x}{1-e^{-x}}\right)^{n+1} e^{mx} x^{-n-1}\, dx$$

$$= \operatorname{Res} \frac{e^{mx}}{(1-e^{-x})^{n+1}}\, dx = \operatorname{Res} \frac{\frac{1}{m} d e^{mx}}{(1-e^{-x})^{n+1}}$$

Let

$$e^x - 1 =: h(x)$$

$$LHS = \operatorname{Res} \frac{\frac{1}{m} d\left(h(x)+1\right)^m}{\left(1-\frac{1}{1+h(x)}\right)^{n+1}} = \frac{(h(x)+1)^{m-1}\, dh(x)}{\left(\frac{h(x)}{1+h(x)}\right)^{n+1}}$$

$$\binom{m+n}{n} = \operatorname{Res} \frac{(z+1)^{m+n}\, dz}{z^{n+1}} \overset{\text{Main Lemma}}{=} \operatorname{Res} \frac{(h(x)+1)^{m+n}\, dh(x)}{h(x)^{n+1}}$$

" 1-line proof ".

Proof of GRR.

Setup $X \subset \mathbb{C}P^n$ X closed compact manifold.

Suppose $\exists$ vector bundle $\mathcal{E} \to \mathbb{C}P^n$ and s a section of $\mathcal{E}$, $s: \mathbb{C}P^n \to \mathcal{E}$ suppose s intersects the zero-section transversally and X is the intersection.

Suppose F is another vector bundle on $\mathbb{C}P^n$, consider $F|_X$. Let us prove GRR for $(X, F|_X)$.

Proof for the setup:

Find a "resolution of $F|_X$ over $\mathbb{C}P^n$".

for instance replace the global sections of $F|_X$ by some complex.

Ex 1 $F = \mathcal{O}(m)$ $m > 0$.

$\Gamma(\mathcal{O}(m)|_X) =$ "homogeneous polynomials of degree m modulo polynomials which vanish on X"

Koszul complex:

$$\ldots \to \Gamma(\mathbb{C}P^n, \mathcal{O}(m) \otimes \mathcal{E}^*) \xrightarrow{Id_{\mathcal{O}(m)} \otimes s} \Gamma(\mathbb{C}P^n, \mathcal{O}(m)).$$

$$\begin{array}{ccc} \mathcal{O}(m) \otimes \mathcal{E}^* & \to & \mathcal{O}(m) \\ f \quad t & & f\langle t, s\rangle \end{array}$$

$x \in \mathbb{C}P^n$ s.t. $s(x) = 0$ $= \langle t(x), s(x)\rangle = 0$.

$s(x) = 0 \iff \forall t \in \mathcal{E}^*_x$ $\langle t, s(x)\rangle = 0$.

homogeneous polynomials which vanish on X = image of the map $\mathcal{O}(m) \otimes \mathcal{E}^* \xrightarrow{s} \mathcal{O}(m)$.

This continues as follows: Let $d = \operatorname{rank} \mathcal{E}$

$K_i := \mathcal{O}(m) \otimes \Lambda^i \mathcal{E}^*$. Let

$K_i \to K_{i-1}$ be the contraction with s.

$$\Lambda^i \mathcal{E}^* \to \Lambda^{i-1} \mathcal{E}^*$$

$$t_1 \wedge t_2 \wedge \ldots \wedge t_i \to \text{Antisymmetrize}(\langle s, t_1\rangle t_2 \wedge \ldots \wedge t_i)$$

results in the

Koszul complex: $K_d \to K_{d-1} \to \ldots \to K_2 \to K_1 \to K_0$.

Claim this complex of vector bundles is exact in all degrees except 0, and $K_0 / \operatorname{Im}(K_1 \to K_0)$ = sections of F over X.

$$\Rightarrow \boxed{\dim(\text{sections of } F \text{ over } X) = \sum_{i=0}^{d} (-1)^i \dim R\Gamma(\mathbb{C}P^n, K_i)}$$

Example $X = 1$ point $\in \mathbb{C}^n$ cut out by equations $x_1 = x_2 = \ldots = x_n = 0$. $\mathcal{E} = \oplus$ of n trivial bundles. $(x_1, \ldots, x_n)$ is a vector-valued section of $\mathcal{E}$. F trivial bundle

$$\ldots \to \text{sections of } \Lambda^2 \mathcal{E}^* \otimes F \to \text{sections of } \mathcal{E}^* \otimes F \to \text{sections of } F$$

$$\ldots \to \text{vector-valued functions on } \mathbb{C}^n \xrightarrow{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} \text{functions on } \mathbb{C}^n.$$

Using GRR for $\mathbb{C}P^n$ we can compute $\dim R\Gamma(\mathbb{C}P^n, K_i)$.

To compute the tangent bundle to X will use a similar complex. ... next time.