1) More on the Euler class
(to complete axiom 3)

2) Splitting principle
(vector bundles ⇒ line bundles)

3) Projective space.

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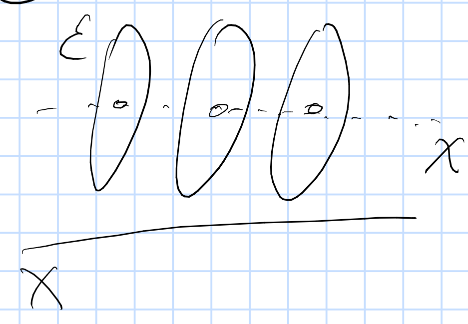
1) Euler class of a (oriented) vector bundle:

$\pi: \mathcal{E} \to X$ complex vector bundle of rank r (locally $\mathbb{C}^r$)

$e(\mathcal{E}) \in H^{2r}(X)$.

Construction: (Milnor–Stasheff) Thom isomorphism.

Let $\mathcal{E}_0 \subset \mathcal{E}$
$\mathcal{E}_0 = \mathcal{E} \setminus X$



Thom class is a canonical class in $\tau \in H^{2r}(\mathcal{E}, \mathcal{E}_0)$ ($X \subset \mathcal{E}$ is a submanifold of codim $2r$)

recall $H^i(\mathcal{E}, \mathcal{E}_0) = 0$ ($i < 2r$).

Note: $X \xrightarrow{0} \mathcal{E} \xrightarrow{\pi} X$ induces iso on cohomology

$$H^i(X) \to H^i(\mathcal{E}) \to H^i(X)$$

mutually inverse.

$$\cup\tau : \underset{\underset{H^i(X)}{\|}}{H^i(\mathcal{E})} \to H^{i+2r}(\mathcal{E}, \mathcal{E}_0)$$

Thom theorem: this is an iso.

e.g. $\underset{\underset{\mathbb{Z}^{\#\text{ conn. comp of } X}}{\|}}{H^0(\mathcal{E})} \xrightarrow{\sim} \underline{H^{2r}(\mathcal{E}, \mathcal{E}_0)}$.

Locally $X = D_d$
$\mathcal{E} = D_d \times \mathbb{C}^r$
homotopy retract to
$X =$ point
$\mathcal{E} = \mathbb{C}^r$
$\mathcal{E}^0 = \mathbb{C}^r \setminus \{0\}$

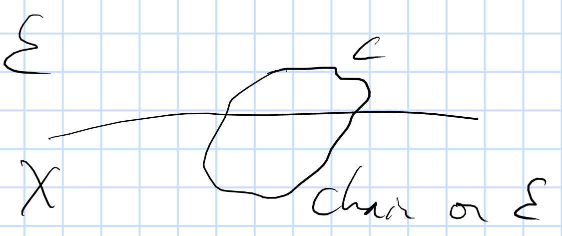
$$H^i(\mathbb{C}^r, \mathbb{C}^r \setminus \{0\}) = \begin{cases} 0 & i \neq 2r \\ \mathbb{Z} & i = 2r \end{cases}$$

Choice of generator of $H^d(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\})$ is a choice of orientation on $\mathbb{R}^d$.

These local Thom classes glue together to a global one

$e(\mathcal{E}) :=$ image of τ under the map $H^{2r}(\mathcal{E}, \mathcal{E}_0) \to H^{2r}(\mathcal{E})$

Recall $H^i(\mathcal{E}, \mathcal{E}_0) \to H^i(\mathcal{E}) \to H^i(\mathcal{E}_0) \to H^{i+1}(\mathcal{E}, \mathcal{E}_0) \to \ldots$.



Chain on $\mathcal{E}$ with boundary in $\mathcal{E}_0$

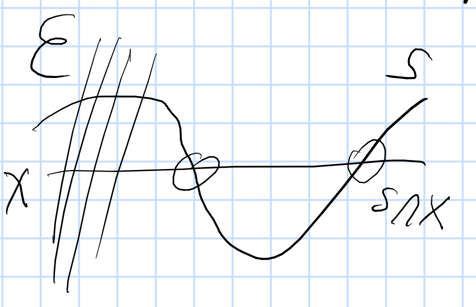
$\iota(c) = \#$ intersection points

more imprecise

Euler class is very useful because:

Let $s : X \to \mathcal{E}$ be a cross-section of $\mathcal{E}$, suppose s intersects $X \subset \mathcal{E}$ transversally.

Then $[s \cap X] = e(\mathcal{E})$.



Why?

$$\begin{array}{ccc} \tau \in H^{2r}(\mathcal{E}, \mathcal{E}_0) & \xrightarrow{f^*} & H^{2r}(\mathcal{E}, \mathcal{E} \setminus s) \\ \downarrow & \boxed{\ } & \downarrow \\ H^{2r}(\mathcal{E}) & \underset{\mathrm{Id}}{\xrightarrow{f^*}} & H^{2r}(\mathcal{E}) \\ \downarrow \wr & & \downarrow \wr \\ e(\mathcal{E}) \in H^{2r}(X) & & H^{2r}(X) \end{array}$$

f^* is induced by
$f : \mathcal{E} \to \mathcal{E}$ $\quad v \in \mathcal{E} \mapsto v - s(\pi(v))$

f homeomorphically maps s to X.

f homotopic to $\mathrm{Id}_{\mathcal{E}}$.
via $v - s(\pi(v)) \cdot t$

$t \in [0, 1]$.

$e(\mathcal{E})$ is obtained by restricting the Thom class from $H^{2r}(\mathcal{E}, \mathcal{E} \setminus s)$ to $H^{2r}(X)$

alternatively:

$$H^{2r}(\mathcal{E}, \mathcal{E} \setminus s) \to H^{2r}(X, X \setminus s \cap X) \to H^{2r}(X)$$

if $s \cap X$ is transversal, then the element in $H^{2r}(X, X \setminus s \cap X)$ is the Thom class of $s \cap X$ in X.

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end of 1) (Thom class) and Euler class.

The zero set of s is $s \cap X$.

The class of $s \cap X$ is independent of s, depends only on $\mathcal{E}$.

Last time: $\mathbb{CP}^1$

tautological line bundle (no sections)

dual tautological bundle (2-dim. space of sections).

More generally, if L is a line bundle on X, then we can consider its tensor powers $L^{\otimes n}$ $n \in \mathbb{Z}$.

dual bundle is L^{-1}.

$$L^{\otimes n} = \underbrace{L \otimes \dots \otimes L}_{n}$$

$L^{\otimes n}$ is constructed by raising all the transition maps to the nth power.

Example on $\mathbb{CP}^1$, L = tautological

On $\mathbb{CP}^1$ 2 charts $\underset{x}{U_1}$, $\underset{y}{U_2}$ with coordinates x, y

on $U_1 \cap U_2$ $y = \frac{1}{x}$

L is given by

$\underset{\mathbb{C}^2}{(x, b)}$ on U_1 $\underset{\mathbb{C}^2}{(y, a)}$ on U_2

over $U_1 \cap U_2$: $y = \frac{1}{x}$, $a = bx$.

$L^{\otimes n}$ is defined similarly using $a = bx^n$.

For $n = -1$ we recover the dual tautological.

Notation $L^{\otimes n} = \mathcal{O}(-n)$, e.g. dual taut. $= \mathcal{O}(1)$. (the one that has sections).

Let's describe the set of holomorphic sections of $\mathcal{O}(n)$. As last time:

on U_1 $b_0 + b_1 x + \dots + b_m x^m$

on U_2 $a_0 + a_1 y + \dots + a_k x^k$

on $U_1 \cap U_2$ $(b_0 + b_1 x + \dots + b_m x^m) x^{-n}$

$= a_k x^{-k} + a_{k-1} x^{-k+1} \dots + a_0$

$\Downarrow$

$m = n = k$

the sections are parametrized by $b_0, \dots, b_n$. dim $= n+1$

	$\dim \Gamma(\mathbb{CP}^1, \mathcal{O}(n))$
$\mathcal{O}(-2)$	0
$\mathcal{O}(-1)$	0
$\mathcal{O}(0) = \mathcal{O}$	1
$\mathcal{O}(1)$	2
$\mathcal{O}(2)$	3

$e(\mathcal{O}(n))$ $(n \geq 0)$

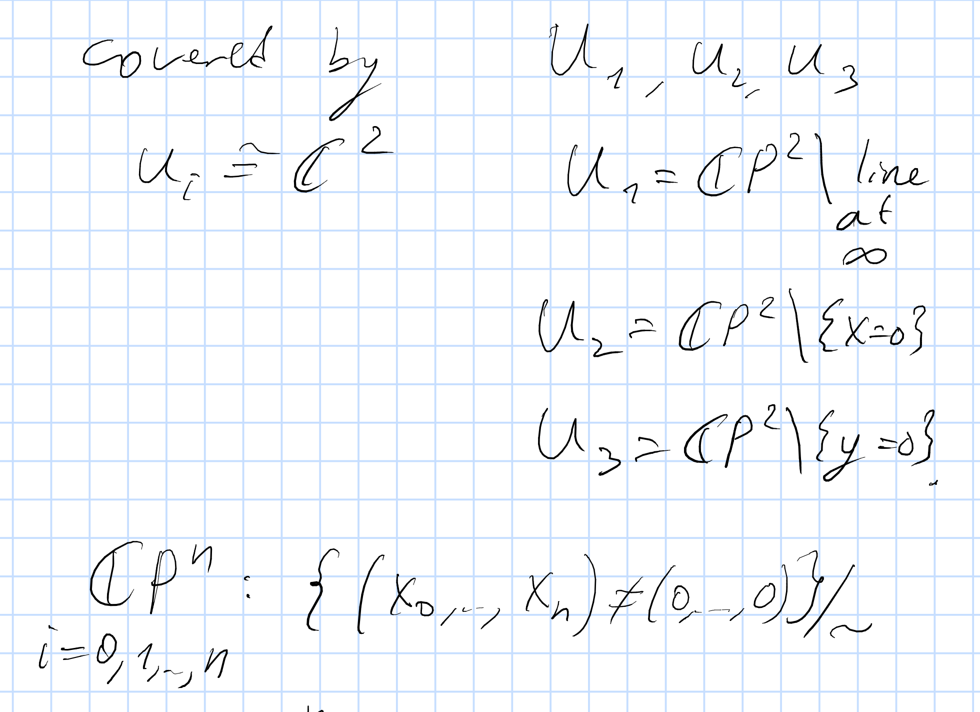
take a section corresponding to a polynomial $b_0 + b_1 x + \dots + b_n x^n$ with n distinct roots $\Rightarrow$

$s \cap X = n$ distinct points

$\Downarrow$

$e(\mathcal{O}(n)) = n \cdot [pt] \in H^2(\mathbb{CP}^1, \mathbb{Z})$

pt = point



covered by U_1, U_2, U_3

$U_i \cong \mathbb{C}^2$

$U_1 = \mathbb{CP}^2 \setminus$ line at ∞

$U_2 = \mathbb{CP}^2 \setminus \{x=0\}$

$U_3 = \mathbb{CP}^2 \setminus \{y=0\}$

$\mathbb{CP}^n$: $\{(x_0, \dots, x_n) \neq (0, \dots, 0)\}/\sim$

$i = 0, 1, \dots, n$

$U_i \subset \mathbb{CP}^n$ given by $x_i \neq 0$.

points on U_i are represented by vectors with $x_i = 1$.

$U_i = \mathbb{C}^n$.

transition functions:

U_i $x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n$

U_j $x'_0, \dots, x'_{i-1}, x'_i, x'_{i+1}, \dots, x'_{j-1}, 1, x'_{j+1}, \dots$

On $U_i \cap U_j$ $x_j \neq 0$ $x'_i \neq 0$

$$\boxed{x'_k = \frac{x_k}{x_j} \qquad x'_i = \frac{1}{x_j}.}$$

Tautological bundle:

$\vec{x} = (x_0, \dots, x_n)$ $(\theta_0, \dots, \theta_n) = \vec{\theta}$ such that $\vec{\theta}$ proportional to $\vec{x}$.

On U_i $x_i = 1$ $\vec{\theta}$ is determined by θ_i. $\forall k$ $\theta_k = x_k \theta_i$.

So, over U_i we have

$\mathbb{C}^n \times \mathbb{C}$

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$(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n, \theta_i)$.

Transition maps:

$(x'_0, \dots, x'_{j-1}, x'_{j+1}, \dots, x'_n, \theta_j)$

$$\boxed{\theta_j = \theta_i \cdot x_j .}$$

Powers of tautological L:

$L^{-m} =: \mathcal{O}(m)$.

Transition maps are given by

$\theta_j = \theta_i \cdot x_j^{-m}$.

Sections $\boxed{x'_k = \frac{x_k}{x_j} \quad x'_i = \frac{1}{x_j}.}$ $| \theta_j = \theta_i \cdot x_j^{-m}$

On U_i a section is given by a polynomial:

$\theta_i = P_i(x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$.

Start with P_0 and put conditions so that P_i is well defined.

$\theta_i = \theta_0 \cdot x_i^{-m} = P_0(x_1, x_2, \dots, x_n) x_i^{-m}$

$\theta_i = P_i(x'_0, x'_1, \dots, x'_{i-1}, x'_{i+1}, \dots, x'_n)$

$= P_i\left(\frac{1}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right)$

$P_i(x'_0, x'_1, \dots, x'_{i-1}, x'_{i+1}, \dots, x'_n) =$

$P_0(x_1, \dots, x_n) x_i^{-m} = P_0\left(\frac{x'_1}{x'_0}, \frac{x'_2}{x'_0}, \dots, \frac{x'_{i-1}}{x'_0}, \frac{1}{x'_0}, \frac{x'_{i+1}}{x'_0}, \dots, \frac{x'_n}{x'_0}\right)(x'_0)^m$ $(*)$

The condition on P_0 is that $(*)$ is a polynomial for any i.

If P_0 contains a monomial $x_1^{c_1} \dots x_n^{c_n}$, then

$(x'_1)^{c_1} \dots (x'_{i-1})^{c_{i-1}} (x'_{i+1})^{c_{i+1}} \dots (x'_n)^{c_n} \times$

$\times x_0^{m - c_1 - c_2 - \dots - c_{i-1} - c_i - \dots - c_n}$

the condition on P is that for all monomials

$$\sum_{i=0}^{n} c_i \leq m.$$