$\dim \mathrm{Hom}_\Gamma$ equivariant Hom between 2 reps $\longleftrightarrow$ $\dim W^\Gamma$ $(W = \mathrm{Hom}(V_1, V_2))$

$\mathrm{Tr}\,\pi$

$$\mathrm{Tr}\,\pi = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} \mathrm{Tr}\, \rho_W(g).$$

$$\dim \mathrm{Hom}_\Gamma(V_1, V_2) = \dim \underbrace{\mathrm{Hom}(V_1,V_2)}_{\text{new }\Gamma\text{-representation}}{}^{\Gamma} \quad \text{= invariants}$$

$$= \frac{1}{|\Gamma|} \sum_{g\in\Gamma} \mathrm{Tr}\left(\rho_{\mathrm{Hom}(V_1,V_2)}(g)\right)$$

last time

To finish the computation we need to solve the problem:

Problem. Let n be fixed. Let A, B be fixed $n\times n$ matrices. Consider the map $\varphi: \mathrm{Mat}_{n\times n} \to \mathrm{Mat}_{n\times n}$ given by $M \to AMB$. Compute the trace of φ.

Warning: not to compute $\mathrm{tr}(AMB)$ (AMB is $n\times n$ matrix $\forall M$)

φ is $n^2\times n^2$ matrix

Answer: $\mathrm{tr}\,\varphi = \mathrm{tr}A\cdot\mathrm{tr}B$.

Proof. $\mathrm{Mat}_{n\times n}$ has basis consisting of elementary matrices $E_{i,j}$. $(E_{i,j})_{i',j'} = \begin{cases} 1 & \text{if } i=i', j=j' \\ 0 & \text{otherwise.}\end{cases}$

$$E_{i,j} = \begin{array}{c|c} & j \\ i & \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \end{array}$$

$$\sum_{i,j=1}^{n} \varphi(E_{i,j})_{i,j} = \mathrm{Tr}\,\varphi.$$

$\varphi(E_{i,j})_{i,j}$ = sum of diagonal entries of φ.

$$(A E_{i,j} B)_{i,j} = \sum_{i',j'} A_{i,i'} (E_{i,j})_{i',j'} B_{j',j}$$

$(i'=i, j'=j)$

$$= A_{i,i} B_{j,j}$$

summing over i,j gives $\mathrm{Tr}\,\varphi = \mathrm{Tr}A\cdot\mathrm{Tr}B$. $\lrcorner$

For our computation:

$\rho(g): M \to \rho_{V_1}(g) M \rho_{V_2}(g)^{-1}$

$\Downarrow$

$\mathrm{tr}\,\rho(g) = \mathrm{Tr}\,\rho_{V_1}(g)\cdot\mathrm{Tr}\,\rho_{V_2}(g^{-1})$, where $\mathrm{Tr}\,\rho_{V_1}(g) = \chi_{V_1}(g)$ and $\mathrm{Tr}\,\rho_{V_2}(g^{-1}) = \chi_{V_2}(g^{-1})$.

Conclusion:

$$\dim \mathrm{Hom}_\Gamma(V_1,V_2) = \frac{1}{|\Gamma|}\sum_g \chi_{V_1}(g)\chi_{V_2}(g^{-1}) \overset{\text{by definition}}{=} \langle \chi_{V_1}, \chi_{V_2}\rangle.$$

So the Formula is proved ∎.

Cor. V is irreducible $\Longleftrightarrow$ $\langle \chi_V, \chi_V\rangle = 1$.

Pf. V irreducible $\Rightarrow$ $\underbrace{\dim \mathrm{Hom}_\Gamma(V,V)}_{=\langle \chi_V,\chi_V\rangle} = 1$ by Schur.

$\Leftarrow$ $\langle \chi_V, \chi_V\rangle = 1$, V is not irreducible

$V = V_1^{m_1}\oplus V_2^{m_2}\oplus \dots \oplus V_k^{m_k}$ $\Rightarrow$

$\chi_V = m_1\chi_{V_1} + \dots + m_k \chi_{V_k}$

$$\langle \chi_V,\chi_V\rangle = \sum m_i m_j (\chi_{V_i},\chi_{V_j}) = \sum m_i m_j \delta_{ij} = \sum m_i^2.$$

V is not irreducible $\Rightarrow$
$m \geq 2 \Rightarrow \sum m_i^2 \geq 2$
or
$m_1 \geq 2 \Rightarrow \sum m_i^2 \geq 4$. contradiction $\lrcorner$

Example 1. For S_3 we had a 2-dimensional rep V with traces $2,0,0,0,-1,-1$. $\langle\chi_V,\chi_V\rangle = \frac{1}{6}\left(2^2+(-1)^2+(-1)^2\right) = 1$.
$\Rightarrow$ V is irreducible!

Example 2. Take $\Gamma = S_n$, V = permutation representation.
V contains $(1,\dots,1)$ invariant vector.
Let $V' = (1,\dots,1)^\perp$.
Claim: V' is irreducible.

Note: $\chi_V = \chi_{V'} + \chi_{\mathbb{C}}$ (trivial rep.)

$\chi_{V'} = \chi_V - \chi_{\mathbb{C}}$

χ_V is easier to compute, in fact

$\chi_V(g) = \mathrm{trace}(\text{permutation matrix of } g)$
$= \#\{i \mid g(i) = i\}$

$$g = \begin{pmatrix} 1\to 2 \\ 2\to 3 \\ 3\to 1\end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{pmatrix} \qquad \mathrm{tr} = 0$$

$\chi_{\mathbb{C}}(g) = 1$

To check: $\sum_{g\in S_n} \left(\#\{i \mid g(i)=i\} - 1\right)^2 = n!$

Home work

Regular representation

$\forall$ finite group Γ it is the permutation representation associated to Γ acting on itself.

explanation:

think $\Gamma = \{g_1, g_2, \dots, g_{|\Gamma|}\}$

each $g\in\Gamma$ gives a permutation π of size $|\Gamma|$ by $g(g_i) = g_{\pi(i)}$.

π gives a permutation matrix of size $|\Gamma|\times|\Gamma|$.

Example. $S_3 = \{\overset{1}{e}, \overset{2}{\tau_{12}}, \overset{3}{\tau_{23}}, \overset{4}{\tau_{13}}, \overset{5}{cyc}, \overset{6}{cyc^{-1}}\}$ (another cycle)

$$\rho(\tau_{12}) = \begin{pmatrix} 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \end{pmatrix}.$$

Let us compute the character of the regular rep of Γ.

Note: $g\neq e \Rightarrow \rho(g)$ has 0 on the diagonal ($gg'\neq g'$).

$$\Rightarrow \chi_{reg}(g) = \begin{cases} 0 & g\neq e \\ |\Gamma| & g = e\end{cases}$$

(reg = regular rep)

Suppose $\mathbb{C}[\Gamma] = V_1^{\oplus m_1}\oplus\dots\oplus V_k^{\oplus m_k}$ is a decomposition into irreducibles.

($\mathbb{C}[\Gamma]$: vector space whose basis is identified with elements of Γ is our notation for the regular rep.)

Q) What is m_i?

A) $m_i = \dim \mathrm{Hom}_\Gamma(V_i, \mathbb{C}[\Gamma])$

$$= (\chi_{V_i}, \chi_{\mathbb{C}[\Gamma]}) = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_{V_i}(g)\chi_{\mathbb{C}[\Gamma]}(g^{-1}) =$$

($\chi_{\mathbb{C}[\Gamma]}(g^{-1}) \neq 0$ only for $g = id$)

$$= \frac{1}{|\Gamma|}\chi_{V_i}(e)|\Gamma| = \chi_{V_i}(e) = \mathrm{Tr}\,\rho_{V_i}(e) = \mathrm{Tr}\,Id_{V_i} = \dim V_i.$$

Conclusion

$$\underbrace{|\Gamma|}_{=\dim\mathbb{C}[\Gamma]} = \sum_{i=1}^{k}(\underbrace{\dim V_i}_{= m_i})^2 \cdot \qquad m_i \cdot \dim V_i$$

Proposition. Let $\mathbb{C}[\Gamma] = V_1^{\oplus m_1}\oplus\dots\oplus V_k^{\oplus m_k}$, V_i irreducible, $V_i\not\cong V_j$ $(i\neq j)$.

Let V' be an irr. rep of Γ. Then $\exists$ unique i such that $V'\cong V_i$.

Pf. $\mathrm{Hom}_\Gamma(\mathbb{C}[\Gamma], V') \neq \{0\}$:

take $v\in V'$, $v\neq 0$.

Let $\varphi: \mathbb{C}[\Gamma]\to V'$ send e_j to $\rho_{V'}(g_j)v$.

[More convenient notation $e_j = [g_j]$

generally $[g]$ = the basis element corresponding to g. $\lrcorner$

Using this notation

$$\varphi([g]) = \rho_{V'}(g)v.$$

To see that $\varphi\in\mathrm{Hom}_\Gamma$

$$\varphi(\rho_{\mathbb{C}[\Gamma]}(g')[g]) = \varphi([g'g]) = \rho_{V'}(g'g)v = \rho_{V'}(g')\rho_{V'}(g)v = \rho_{V'}(g')\varphi([g]).$$

Clearly $\varphi\neq 0$ ($\varphi([e]) = v\neq 0$).

So $\mathrm{Hom}_\Gamma(\mathbb{C}[\Gamma], V')\neq\{0\}$.

In the decomposition $\mathbb{C}[\Gamma] = V_1^{\oplus m_1}\oplus\dots$

$\varphi: \mathbb{C}[\Gamma]\to V'$ $\neq 0$

$\Rightarrow \exists$ direct summand where φ is not 0. Say V_i

restricting to this summand we obtain an element of $\mathrm{Hom}_\Gamma(V_i, V')$ which is $\neq 0$. $\Rightarrow$ $V'\cong V_i$ by Schur ∎

Conclusion

Each irreducible representation V of Γ appears in the decomposition of $\mathbb{C}[\Gamma]$ as many times as $\dim V$.

Problem of classifying irreducible reps is solved by constructing irreducible reps $V_1, V_2, \dots$ until $\sum \dim V_i^2 = |\Gamma|$.

Ex 0. Cyclic group $\mathbb{Z}/n\mathbb{Z} = \Gamma$

1-dim representations $\forall k = 0,1,\dots,n-1$

$\rho(i) = e^{\frac{2\pi\sqrt{-1}}{n}ik}$.

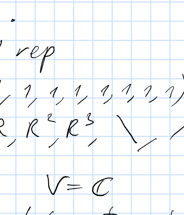
So we constructed n reps,

$|\Gamma| = n = \sum_k 1^2 = n$.

Ex 1. $\Gamma = S_3$ $\quad |\Gamma| = 3! = 6$.

we constructed 3 non-isomorphic irreducible reps: trivial, sign, 2-dim. $1^2+1^2+2^2 = 6$, so these are all.

Ex 2. D_4



D_n = symmetries of the regular n-gon.

reflections, rotations

rotation · rotation = rotation
reflection · rotation = reflection
reflection · reflection = rotation

Reflections: \ / | —

Rotations: $90° = R$, R^2, R^3, $R^4 = e$.

8 elements.

trivial rep

$\chi_{triv} = (1,1,1,1,1,1,1,1)$

$\Gamma = (e, R, R^2, R^3, \diagdown, \diagup, |, —)$

Sign rep $V = \mathbb{C}$

$\rho(g) = \det \in \pm 1$ depending on g preserves the orientation or not.

$\chi_{sign} = (1,1,1,1,-1,-1,-1,-1)$

Standard 2-dimensional rep

$\rho_{st}(g) = g$.

$\chi_{st} = (2, 0, -2, 0, 0, 0, 0, 0)$

$(\chi_{st}, \chi_{st}) = 1 \Rightarrow st$ is irreducible

$8 - 1^2 - 1^2 - 2^2 = 2$

need 2 more 1-dim reps. homework.

Ex 3. Homework S_4 $\quad 4! = 24$

$24 = 1^2 + 1^2 + 3^2 + 3^2 + 2^2$.