Character table: rows are irreducible representations, columns are conjugacy classes

conj. classes

$$\begin{array}{c} \\ \text{reps} \end{array} \begin{pmatrix} \\ \chi_{V_i}(g_j) \\ \\ \end{pmatrix}$$

$V_1, \dots, V_m$ are reps

$g_1, \dots, g_{m'}$ are representatives of conjugacy classes

1st orthogonality relation: between 2 reps $V_i, V_{i'}$ based on

$$\dim \mathrm{Hom}_\Gamma (V_i, V_{i'}) = \begin{cases} 0 & i \neq i' \\ 1 & i = i' \end{cases}$$

computing this dimension in terms of characters produced the relation.

2ud orthogonality: Pick conjugacy classes C, C'. Consider $g \in C$ $g' \in C'$ if $g = (g')^{-1}$ then

$$\rho_{\mathbb{C}[\Gamma]}(g)\rho_{\mathbb{C}[\Gamma]}(g') = \rho_{\mathbb{C}[\Gamma]}(e) = \mathrm{Id}$$

otherwise — " — $= \underbrace{\rho_{\mathbb{C}[\Gamma]}(\overset{\neq e}{gg'})}_{\text{a permutation matrix of a permutation without fixed points.}}$

notice:

$$\mathrm{tr}\left(\rho_{\mathbb{C}[\Gamma]}(g)\rho_{\mathbb{C}[\Gamma]}(g')\right) = \begin{cases} |\Gamma| & g = g'^{-1} \\ 0 & \text{otherwise} \end{cases}$$

hence

$$\sum_{g \in C} \sum_{g' \in C'} \mathrm{tr}\left(\rho_{\mathbb{C}[\Gamma]}(g)\rho_{\mathbb{C}[\Gamma]}(g')\right) = \begin{cases} 0 & C \neq C'^{-1} \\ |\Gamma| \cdot |C| & C = C'^{-1} \end{cases}$$

Rewrite LHS as follows:

$$\mathrm{tr}\left(\sum_{g \in C} \rho_{\mathbb{C}[\Gamma]}(g) \cdot \sum_{g' \in C'} \rho_{\mathbb{C}[\Gamma]}(g')\right).$$

Use the decomposition

$$\mathbb{C}[\Gamma] = V_1^{\oplus \dim V_1} \oplus V_2^{\oplus \dim V_2} + \dots$$

computing trace in $\mathbb{C}[\Gamma]$ is the same as computing traces in each V_i and then adding up with coefficient $\dim V_i$.

$$LHS = \sum_{i=1}^m \dim V_i \, \mathrm{tr}\left(\sum_{g \in C} \rho_{V_i}(g) \sum_{g' \in C'} \rho_{V_i}(g')\right)$$

Observe: $\sum_{g \in C} \rho_{V_i}(g)$ commutes with $\rho_{V_i}(h)$ $\forall h \in \Gamma$:

$$\sum_{g \in C} \rho_{V_i}(g)\rho_{V_i}(h) = \sum_{g \in C} \rho_{V_i}(gh)$$

$$\rho_{V_i}(h)\sum_{g \in C} \rho_{V_i}(g) = \sum_{g \in C} \rho_{V_i}(h \cdot \underbrace{h^{-1}gh}_{\in C})$$

$g \to h^{-1}gh$ is a bijection $C \to C$

by Schur's lemma $\sum_{g \in C} \rho_{V_i}(g)$ is a scalar matrix. $= \mathrm{const} \cdot \mathrm{Id}$.

$$\mathrm{tr} \sum_{g \in C} \rho_{V_i}(g) = \mathrm{const} \cdot \dim V_i$$

so

$$\sum_{g \in C} \rho_{V_i}(g) = \frac{1}{\dim V_i}\left(\mathrm{tr} \sum_{g \in C} \rho_{V_i}(g)\right) \cdot \mathrm{Id}_{V_i} = \frac{|C|}{\dim V_i} \chi_{V_i}(g) \cdot \mathrm{Id}_{V_i}$$

where g is any representative of C.

so

$$\mathrm{tr} \sum_{g \in C} \rho_{V_i}(g) \sum_{g' \in C'} \rho_{V_i}(g') = \dim V_i \cdot \frac{|C|}{\dim V_i} \chi_{V_i}(g) \frac{|C'|}{\dim V_i} \chi_{V_i}(g'),$$

where g, g' are representatives of C, C'.

$$LHS = \sum_{i=1}^m \chi_{V_i}(g) \chi_{V_i}(g') \cdot |C| \cdot |C'|$$

Conclusion

$$\sum_{i=1}^m \chi_{V_i}(g) \chi_{V_i}(g') = \begin{cases} 0 & \text{if } g, g'^{-1} \text{ are in different conjugacy classes} \\ \frac{|\Gamma|}{|C|} & \text{if in the same.} \end{cases}$$

$$\sum_{i=1}^m \chi_{V_i}(g_j) \chi_{V_i}(g_{j'}) = \begin{cases} 0 & \text{if } j \neq \bar{j'} \\ \frac{|\Gamma|}{|C_j|} & \text{if } j = \bar{j'} \end{cases}$$

$\bar{j}$ is the number of the conjugacy class C_j^{-1} $(C_j^{-1} = C_{\bar{j}})$.

Idea: Orthogonality implies linear independence:

Suppose $\sum_{i=1}^{m} \alpha_i r_i = 0$, where

r_i = i-th row $(r_i = \chi_{V_i})$

then

$$0 = \left(\sum_{i=1}^{m} \alpha_i r_i , r_{i_0} \right) = \alpha_{i_0} \quad \forall i_0$$

$\Rightarrow$ r_i are linearly independent.

Similarly for the columns:

$$\sum_{i=1}^{m'} \alpha_j c_j = 0$$

$$0 = \left(\sum \alpha_j c_j , c_{j_0} \right) = \alpha_{j_0} \quad \forall j_0$$

$\Rightarrow$ columns are linearly independent.

$\Rightarrow$ rank $M = m$, rank $M = m'$ $\Rightarrow m = m'$!

The conclusion

<u>Theorem</u> $\forall$ finite group Γ the number of irreducible representations equals to the number of conjugacy classes.

<u>Remark</u> Can we have a bijection between these? Usually no nice bijection exists.

Break: $\boxed{10:33}$

Quiz: (Answers by private message in the chat)

Ex 1

1) List the sizes of the conjugacy classes of the symmetry group of the square

2) List the the dimensions of the irreducible representations.

15 minutes until 10:55.

Ex 2 the same for S_3

Ex 3 S_4

Ex 4 S_5

Answers

$D4$	1, 1, 2, 2, 2	1, 1, 1, 1, 2
S_3	1, 2, 3	1, 1, 2
S_4	1, 6, 3, 6, 8	1, 1, 3, 3, 2
S_5	1, 10, 15, 20, 20, 24, 30	1, 1, 4, 4, 5, 5, 6

id, 2,1,1,1, 2,2,1, (3,2), (3,1,1), (5), (4,1)

# of permutations of cycle type $(a_1, \dots, a_k)$ $\quad a_1 \geq a_2 \geq \dots \geq a_k$

$$\frac{(\sum a_i)!\ \prod (a_i - 1)!}{\prod a_i!\ \prod m_i!} = \frac{(\sum a_i)!}{\prod a_i \prod m_i!}$$

$n = \sum a_i$

$m_i = \#\, i$ in $a_1, \dots, a_k$

S_5: $2,2,1$: $\frac{120}{4 \cdot 2} = 15$

$2,3$: $\frac{120}{2 \cdot 3} = 20$ $\quad 4,1$: $\frac{120}{4} = 30$

$3,1,1$: $\frac{120}{3 \cdot 2} = 20$

Question is there any connection between 2 lists? Probably not.

$\sum |C_i| = |\Gamma|$

$\sum (\dim V_i)^2 = |\Gamma|$

$|C_i|$ divides the order $|\Gamma|$.

why? $|\Gamma| = |C_i| \cdot |Z_i|$

$Z_i = \{ h \in \Gamma : h g_i = g_i h \}$

Less trivial:

$\dim V_i$ divides the size of $|\Gamma|$.

Why?