Today: <u>Maschke's Teorem</u>

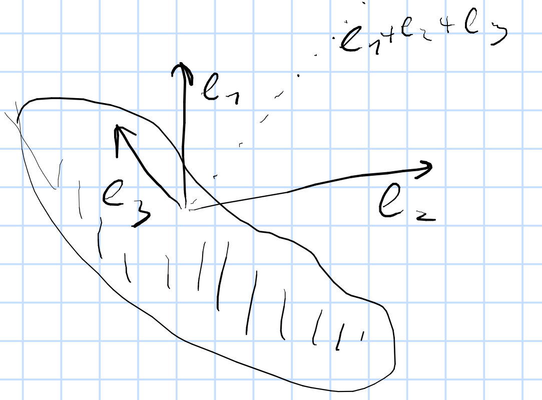
Before, look more on S_3, 2-dim rep.

Take 3-dim rep with basis e_1, e_2, e_3

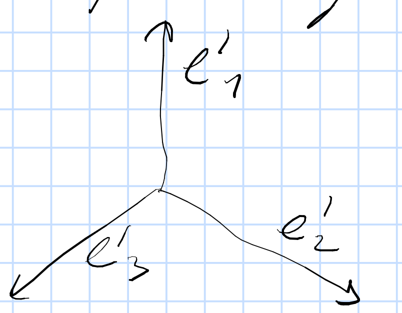
action permutes $\rho(g)(e_i) = e_{g(i)}$

$g \in S_3 \quad i = 1,2,3.$

we obtained the 2-dim rep by taking $(e_1+e_2+e_3)^\perp$



Project everything to the plane:



$e_i' = \mathrm{proj}(e_i)$

$e_1' + e_2' + e_3' = 0$

S_3 permutes the vertices of the equilateral triangle

Exercise: write down $\rho(g)$ as a 2×2 matrix

Let's use e_1', e_2' as a basis

$e = \begin{pmatrix} 1\to 1 \\ 2\to 2 \\ 3 \to 3\end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$ $\qquad \tau_{13} = \begin{pmatrix} 1\to 3 \\ 2\to 2 \\ 3 \to 1\end{pmatrix} \dashrightarrow \begin{pmatrix} -1 & 0 \\ -1 & 1\end{pmatrix}$

$\tau_{12} = \begin{matrix} 1\to 2 \\ 2\to 1 \\ 3 \to 3\end{matrix} \dashrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$ $\qquad$ cycle $\begin{pmatrix} 1\to 2 \\ 2\to 3 \\ 3 \to 1\end{pmatrix} \dashrightarrow \begin{pmatrix} 0 & -1 \\ 1 & -1\end{pmatrix}$

$\tau_{23} = \begin{pmatrix} 1\to 1 \\ 2\to 3 \\ 3 \to 2\end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & -1 \\ 0 & -1\end{pmatrix}$ $\qquad$ another cycle $\begin{pmatrix} 1\to 3 \\ 2\to 1 \\ 3 \to 2\end{pmatrix} \dashrightarrow \begin{pmatrix} -1 & 1 \\ -1 & 0\end{pmatrix}$

check:

$\tau_{23}\,\text{cycle} = \begin{pmatrix} 1\to 2\to 3 \\ 2\to 3\to 2 \\ 3\to 1\to 1\end{pmatrix} = \tau_{13}$ $\qquad \underbrace{\begin{pmatrix} 1 & -1 \\ 0 & -1\end{pmatrix}}_{\tau_{23}} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & -1\end{pmatrix}}_{\text{cycle}} = \underbrace{\begin{pmatrix} -1 & 0 \\ -1 & 1\end{pmatrix}}_{\tau_{13}}$

composition

traces:

	e	τ_{12}	τ_{23}	τ_{13}	cycle	another cycle
trace in 2-dim rep	2	0	0	0	-1	-1
trivial rep	1	1	1	1	1	1
Sign rep	1	-1	-1	-1	1	1

<u>Character table</u>

More representations

* trivial representation (for any group $V = \mathbb{C}$ $\rho_V(g) = I_1$ $\forall g \in \Gamma$)

* Sign representation (for $\Gamma = S_n$ $V = \mathbb{C}$ $\rho_V(g) = \begin{cases} I_1 & g \text{ is even} \\ -I_1 & g \text{ is odd}\end{cases}$

recall even · even = even
even · odd = odd $\Rightarrow$ we get a representation
odd · odd = even)

Interesting observation: rows are orthogonal! We will prove it in general.

Now: Maschke's Teorem

Recall

Def 1 (V,ρ_V) is a rep. of Γ, we say it is **decomposable** if $\exists\, V_1 \subset V,\ V_2 \subset V$ linear subspaces such that $V_1 \neq \{0\},\ V_2 \neq \{0\},\quad V = V_1 + V_2,\ V_1 \cap V_2 = \{0\}$,
$\forall g \in \Gamma\quad \rho_V(g)(V_1) \subset V_1 \quad \rho_V(g)(V_2) \subset V_2$.

Exercise Prove that this is equivalent to $V \cong V_1 \oplus V_2$
(Lecture 1)

Def 2 (V,ρ_V) is a rep of Γ, we say it is **reducible** if $\exists\, V_1 \subset V$ linear subspace
$V_1 \neq \{0\},\ V_1 \neq V,\quad \forall g \in \Gamma\ \rho_V(g)(V_1) \subset V_1$.

Conversely, if (V,ρ_V) is not decomposable, it is called indecomposable

—— || —— is not reducible, —||— irreducible or simple.

Let us compare these 2 notions

decomposable $\Rightarrow$ reducible
irreducible $\Rightarrow$ indecomposable

Question is converse true?

Example 1

group Γ = invertible upper-triangular 2×2 matrices $V = \mathbb{C}^2$

$\forall g \in \Gamma \qquad \rho_V(g) = g$.

denote basis e_1, e_2. $\mathrm{Span}(e_1)$ is an invariant subspace

$g = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \Gamma \qquad g(e_1) = a\, e_1$, so $g(\mathrm{span}(e_1)) \subset \mathrm{span}(e_1)$

so it is reducible.

it is not decomposable! Otherwise we have V_1, V_2 $\dim V_1 = \dim V_2 = 1$. Choose a basis of $V_1 =: e_1$ of $V_2 =: e_2$. In this new basis every $\rho_V(g)$ is diagonal. $\Rightarrow \rho_V(g_1)\rho_V(g_2) = \rho_V(g_2)\rho_V(g_1)$
$\forall g_1, g_2 \in \Gamma$. Contradiction:

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ does not commute with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Example 2 $\qquad \Gamma = \mathbb{Z} \qquad V = \mathbb{C}^2$

$\rho_V(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \forall n \in \mathbb{Z}\quad \rho_V(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

Again, $\mathrm{span}(e_1)$ is an invariant subspace.

If we had $V = V_1 \oplus V_2 \Rightarrow \rho_V(n)$ is in some basis diagonal, since eigenvalues are $1 \Rightarrow$
$\rho_V(n) = \mathrm{Id}_V$ contradiction.

so reducible ✓
decomposable ✗.

After break: Proof for finite groups that
Def 1 $\Leftrightarrow$ Def 2.

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Theorem Suppose Γ is a finite group, (V, ρ_V) is a f.d. representation, $V_1 \subset V$ an invariant subspace, i.e. $\rho_V(g) V_1 \subset V_1$. There exists V_2 also invariant, $V_1 \cap V_2 = \{0\}$, $V = V_1 + V_2$.

Corollary if Γ is finite, then reducible $\Rightarrow$ decomposable.

Pf. Main idea use things like $\sum_{g \in \Gamma}$.

Begin with $V_1 \subset V$. Choose any V_2 such that $V_1 \cap V_2 = \{0\}$, $V = V_1 + V_2$. For example choose basis for V_1 $e_1, e_2, \ldots, e_k$, complete it to a basis of V $e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_n$, Let $V_2 = \mathrm{span}(e_{k+1}, \ldots, e_n)$.

We plan to use the main idea to find another V_2' s.t. V_2' is invariant.

Let π be the projection to V_1 along V_2:

$\pi(x) = x \quad x \in V_1$
$\pi(x) = 0 \quad x \in V_2$.

Let $\tilde{\pi} = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \rho(g) \pi \rho(g)^{-1}$.

Properties: 1) $\forall g \in \Gamma \quad \rho(g)\tilde{\pi} = \tilde{\pi}\rho(g)$:

$$\rho(g)\tilde{\pi} = \frac{1}{|\Gamma|} \rho(g) \sum_{g' \in \Gamma} \rho(g') \pi \rho(g')^{-1} = \frac{1}{|\Gamma|} \sum_{g' \in \Gamma} \rho(gg') \pi \rho(g')^{-1} =$$

order of Γ

$$\underset{gg' = g''}{=} \frac{1}{|\Gamma|} \sum_{g'' \in \Gamma} \rho(g'') \pi \rho(g^{-1}g'')^{-1} = \tilde{\pi}\rho(g).$$

$$(\rho(g)^{-1}\rho(g''))^{-1} = \rho(g'')^{-1}\rho(g).$$

2) $\mathrm{Im}\,\tilde{\pi} = V_1$:

$\Rightarrow$ $x \in \mathrm{Im}\,\tilde{\pi} \Rightarrow x = \tilde{\pi}(y) \Rightarrow x = \frac{1}{|\Gamma|} \sum \rho(g) \underbrace{\pi \rho(g)^{-1}(y)}_{\in V_1} \Rightarrow$

each summand $\in V_1$ $\Rightarrow x \in V_1$

$\Leftarrow$ $x \in V_1 \Rightarrow \tilde{\pi}(x) = \frac{1}{|\Gamma|} \sum \rho(g) \pi \underbrace{\rho(g)^{-1}(x)}_{\in V_1}$ $\quad (*)$

$$= \frac{1}{|\Gamma|} \sum \rho(g)\rho(g)^{-1}(x) = \frac{|\Gamma|}{|\Gamma|} x = x.$$

$\Rightarrow x \in \mathrm{Im}\,\tilde{\pi}$.

Take $\tilde{V}_2 = \mathrm{Ker}\,\tilde{\pi}$. Let us prove that $V_1, \tilde{V}_2$ satisfy Def 1.

1) $\tilde{V}_2$ is invariant: $x \in \tilde{V}_2 \Rightarrow \tilde{\pi}x = 0 \Rightarrow \tilde{\pi}\rho_V(g)x = \rho_V(g)\tilde{\pi}x = 0$

$\Rightarrow \rho_V(g)x \in \tilde{V}_2$

2) $\tilde{V}_2 \cap V_1 = \{0\}$: $x \in V_1 \cap \tilde{V}_2 \Rightarrow \tilde{\pi}x = 0 \quad (x \in \tilde{V}_2)$

$\tilde{\pi}x = x \quad (x \in V_1)$ by $(*)$

$\Rightarrow x = 0$.

3) $V = V_1 + \tilde{V}_2$

Let $x \in V \Rightarrow \tilde{\pi}(x) \in V_1 \Rightarrow \tilde{\pi}^2(x) = \tilde{\pi}(x) \Rightarrow$

$\tilde{\pi}(x - \tilde{\pi}(x)) = 0 \Rightarrow x - \tilde{\pi}(x) \in \tilde{V}_2$

$$x = \underset{V_1}{\tilde{\pi}(x)} + \underset{\tilde{V}_2}{x - \tilde{\pi}(x)}.$$

$\square$

Remarks 1) For infinite dimensional rep the proof still works. Why does V_2 exist?

Consider V_2 s.t. $V_2 \cap V_1 = \{0\}$, by Zorn's lemma $\exists$ maximal such V_2.

$\Rightarrow V = V_1 + V_2$.

If you want for Banach spaces V_1, V_2 closed. probably proof won't work.

2) What if our base field has finite characteristic? statement is false! (Still true if char $\nmid |\Gamma|$).

Ex over $\mathbb{Z}/p\mathbb{Z}$ $\quad \Gamma = \mathbb{Z}/p\mathbb{Z}$

$$\rho_V(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \rho_V(p) = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\forall k \in \mathbb{Z}/p\mathbb{Z} \quad \rho_V(k) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ · reducible, not decomposable.