Exam: 26.06.2020 (written)

$E_{12}^{a}E_{23}^{c} = \begin{pmatrix} 1 & a & ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = E_{23}^{c}E_{12}^{a}E_{13}^{ac}$

we computed last time: p^2 1-dim reps ← clear

$p-1$ p-dim reps ← find these

Observation: E_{13} is central, i.e. it commutes with the rest.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} = \begin{matrix} 1 & a & b+1 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{matrix} = \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}.$$

So in any irreducible rep E_{13} is a scalar matrix $(\lambda \cdot I)$ $\lambda \in \mathbb{C}$.

Notice E_{13} acted as 1 in all 1-dimensional reps.

$(E_{12}E_{23} = E_{23}E_{12}E_{13})$.

So in some of these p-dimensional reps E_{13} must be not equal to I. $E_{13}^{p} = I \Rightarrow$

E_{13} must act as $e^{\frac{2\pi i}{p}k}I$ $k = 1, 2, \dots, p-1$.

In fact, let's assume $\rho(E_{13}) = e^{\frac{2\pi i}{p}k}I$, construct it.

For each k we will obtain a p-dim rep.

Let $\varsigma = e^{\frac{2\pi i}{p}k}$

$$\rho(E_{13}) = \begin{pmatrix} \varsigma & & 0 \\ & \varsigma & \\ 0 & & \ddots\ \varsigma \end{pmatrix}. \quad \text{Assume } \rho(E_{12}) = \begin{pmatrix} 0 & 0 & & & 1 \\ 1 & 0 & & & \\ 0 & 1 & 0 & & 0 \\ & & & & 0 \\ 0 & & 0 & 1 & 0 \end{pmatrix}$$

$\rho(E_{13})e_i = \varsigma e_i$ $\quad \rho(E_{12})e_i = e_{i+1}$

$\rho(E_{12})e_p = e_1$

$\rho(E_{23})e_i = c_i e_i$ $\quad c_i \in \mathbb{C}$

$E_{12}E_{23} = E_{23}E_{12}E_{13}$ $(*)$

$c_i e_{i+1} = \varsigma c_{i+1} e_{i+1}$

$\Downarrow$

Take $c_1 = 1$, $c_2 = \varsigma^{-1}$, … $c_p = \varsigma^{1-p}$, $c_1 = \varsigma^{-p} = 1$.

$$\rho(E_{23}) = \begin{pmatrix} 1 & & 0 \\ & \varsigma^{-1} & \\ 0 & & \varsigma^{-2} \\ & & & \ddots \end{pmatrix}$$

Check $(*)$, $\rho(E_{12})^{p} = I$, $\rho(E_{23})^{p} = I$, $\rho(E_{13})$ is central. These are enough.

Character table:

Conjugacy classes:

$e, E_{13}, \dots, E_{13}^{p-1}$, $\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ $(a,b) \in (\mathbb{Z}/p\mathbb{Z})^2 \setminus \{0,0\}$

clearly are in distinct conjugacy classes.

obtain $\geq p^2 + p - 1$ conjugacy classes.

since we know # of irreps $= p^2 + p - 1$, this is a complete list.

	1-dim rep $\begin{pmatrix} 1 & a & \\ & 1 & b \\ & & 1 \end{pmatrix} \to e^{\frac{2\pi i}{p}(am+bn)}$	p-dim rep $E_{13} \to e^{\frac{2\pi i}{p}k}$
e	1	p
E_{13}	1	$p e^{2\pi i/p k}$
$\vdots$		
E_{13}^{p-1}	1	$p e^{\frac{2\pi i}{p} \cdot k(p-1)}$
$\begin{pmatrix} 1 & a & 0 \\ & 1 & b \\ & & 1 \end{pmatrix}$	$e^{\frac{2\pi i}{p}(am+bn)}$	0
	p^2 columns (m,n)	$p-1$ columns k

Ex check orthogonality.

Conjecture (Higman): Fix n. Let $U_n(p)$ be the group of $n \times n$ upper-triangular matrices over $\mathbb{Z}/p\mathbb{Z}$ with 1 on the diagonal, number of conjugacy classes in $U_n(p)$ is a polynomial of p. (fixed n).

(Pak: true for n small, false for n large (probably)).

Symmetric groups

$\Gamma = S_4$ $\quad |S_4| = 24 = 1^2 + 3^2 + 2^2 + 3^2 + 1^2$.

1-dimensional reps: triv, sign.

3-dimensional reps: $(1,1,1,1)^{\perp} \subset \mathbb{C}^4$ ← permutation rep'n

Def a representation is faithful if any element $g \in \Gamma \setminus \{e\}$ acts by a matrix different from identity. — this one is faithful.

Construction: (V, ρ_V) a representation, (W, ρ_W) a 1-dim representation $\rho_W(g) \in \mathbb{C}$

A new representation $(V \otimes W, \rho_{V\otimes W})$ is defined as a vector space: V, $\rho_{V\otimes W}(g) = \rho_V(g) \cdot \underset{\mathbb{C}}{\rho_W(g)}$.

another 3-dim rep is standard $\otimes$ sign.

How to get the 2-dim rep?

Generally: Pick numbers $\lambda_1, \lambda_2, \dots, \lambda_k$ $\sum_{i=1}^{k} \lambda_i = n$ (a partition).

Use λ_1 letters "a", λ_2 letters "b", …:

$\underbrace{a \dots a}_{\lambda_1}\underbrace{b \dots b}_{\lambda_2} \dots$ obtained a word of n letters. permutations permute the letters of this word

→ an action of S_n on the set of words of length n consisting of λ_1 letters "a", λ_2 letters "b" and so on.

Let $\mathrm{Ind}_{\lambda_1, \dots}$ denote the permutation representation corresponding to this action.

examples: $(n=4)$ $\quad \lambda = (4)$ $\lambda_1 = 4, k = 1$. Only 1 word aaaa

$\mathrm{Ind}_{(4)} = $ trivial.

$\lambda = (3,1)$ aaab aaba abaa baaa

$\mathrm{Ind}_{(3,1)} = \text{standard} \oplus \text{trivial}$.

$\lambda = (2,2)$ aabb abab baab abba baba bbaa.

6-dimensional → 5 dimensional if we subtract trivial $5 = 3 + 2$. hope 2 is the missing 2-dim rep.

Let's identify words which are obtained by exchanging a and b.

→ obtain a 3-dimensional permutation rep, subtract trivial, get 2-dimensional rep.

Compute character table:

	triv	sign	std	std⊗sign	2-dimensional	size of conj class
$(1,1,1,1)$	1	1	3	3	2	1
$(2,1,1)$	1	−1	1	−1	0	6
$(3,1)$	1	1	0	0	−1	8
$(2,2)$	1	1	−1	−1	2	3
(4)	1	−1	−1	1	0	6

conjugacy classes (list lengths of cycles)

= product 2nd and 3rd

$2^2 + (-1)^2 \cdot 8 + 2^2 \cdot 3 = 24$

Break 10:55

$\mathrm{Ind}_{2,2} = \text{2-dim} \oplus \text{standard} \oplus \text{trivial}$

	$1 \leftrightarrow 2$	$1 \to 2 \to 3 \to 1$	$1 \leftrightarrow 2$, $3 \leftrightarrow 4$	$1 \to 2 \to 3 \to 4 \to 1$
a a b b	fixed		fixed	
a b a b			fixed	fixed
a b b a			fixed	

Theorem Fix n, list partitions of n $\lambda = (\lambda_1 \geq \lambda_2 \dots)$ in reverse lexicographic order. $\exists$ irreducible representations of S_n S_λ, so that $\mathrm{Ind}_\lambda = S_\lambda \oplus$ a combination of S_μ where μ are before in the list.

(n) $\quad \mathrm{Ind}_{(n)}$

$(n-1, 1)$ $\quad \mathrm{Ind}_{(n-1,1)}$

$(n-2, 2)$ $\quad \mathrm{Ind}_{(n-2,2)}$

$(n-2, 1, 1)$ $\quad \vdots$

$\vdots$

$(1, 1, \dots 1)$

For instance $n = 4$

$\mathrm{Ind}_{(4)} = S_{(4)} = $ trivial

$\mathrm{Ind}_{(3,1)} = \underset{\text{new}}{S_{(3,1)}} + \underset{\text{before}}{S_{(4)}}$

$\mathrm{Ind}_{(2,2)} = \underset{\text{new}}{S_{(2,2)}} + \underbrace{S_{(3,1)} + S_{(4)}}$ $\quad$ Kostka numbers

$\mathrm{Ind}_{(2,1,1)}$ (12-dimensional) $= S_{(2,1,1)} + S_{2,2} + 2S_{3,1} + S_4$

aabc $\quad S_{(2,1,1)} = S_{(3,1)} \otimes \text{sign}$

$3 + 2 + 2 \cdot 3 + 1 = 12$

$\mathrm{Ind}_{(1,1,1,1)}$ (24-dimensional) $= S_{(1,1,1,1)} + 3 \cdot S_{(2,1,1)} + 2S_{(2,2)} + 3S_{(3,1)} + S_{(4)}$.

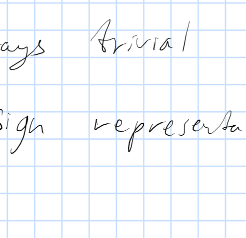
abcd $\quad$ (24-dimensional) $= \mathbb{C}[S_4]$, $\quad S_{(1,1,1,1)} = $ Sign

S_λ can be constructed as the orthogonal complement to all the old representations in Ind_λ.

This theorem can be proven by orthogonality relations.

$S_{2,1,1} \to$

$S_{3,1} \to$



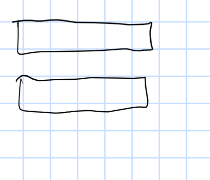
$\otimes$ sign.

$S_{(n)}$ always trivial

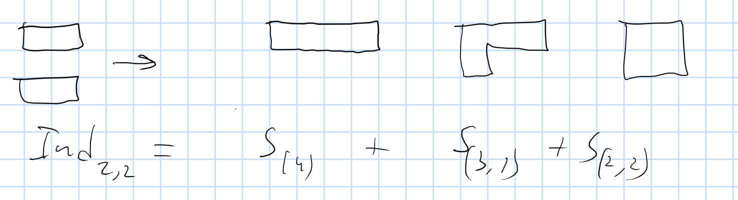
$S_{(1,1 \dots 1)}$ Sign representation. (doesn't appear in any Ind except $\mathrm{Ind}_{1 \dots 1} = \mathbb{C}[S_n]$).

How many times S_μ appears in Ind_λ?

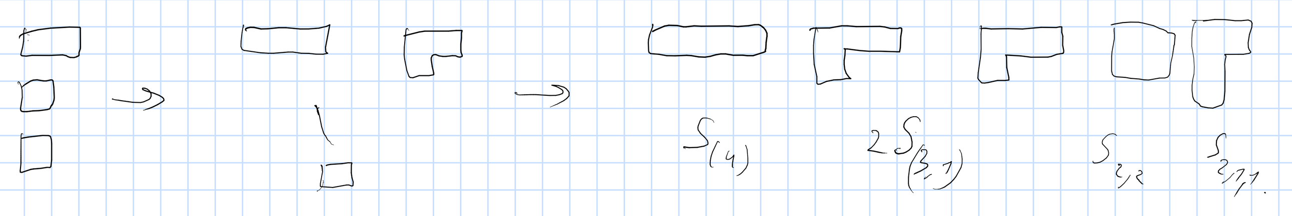
Start by λ_1 λ_2 λ_3



each move you split λ_i into parts and add to the picture:



$\mathrm{Ind}_{2,2} = S_{(4)} + S_{(3,1)} + S_{(2,2)}$



$S_{(4)}$ $\quad 2S_{(3,1)}$ $\quad S_{2,2}$ $\quad S_{2,1,1}$.