Last time: Maschke's theorem
(irreducible $\iff$ indecomposable)

Today: Schur's lemma,
deduce properties of characters
helping to classify all representations
of a finite group.

Notation: $(V_1, \rho_{V_1}) \xleftarrow{\Gamma\text{-reps}} (V_2, \rho_{V_2})$

- $Hom(V_1, V_2)$ = space of linear maps from V_1 to V_2.

e.g. $V_1 = \mathbb{C}^m$ $V_2 = \mathbb{C}^n$ $Hom(V_1, V_2)$ = $n \times m$ matrices, so

$$\dim Hom(V_1, V_2) = \dim V_1 \dim V_2$$

- $Hom_\Gamma(V_1, V_2) = \{f: V_1 \to V_2 \mid \forall g \in \Gamma \ \forall x \in V_1 : f(\rho_{V_1}(g)x) = \rho_{V_2}(g) f(x)\}$

$Hom_\Gamma(V_1, V_2) \subset Hom(V_1, V_2)$

Schur's lemma:
Suppose (V_1, ρ_{V_1}), (V_2, ρ_{V_2}) are finite dim irreducible reps of a group Γ, then

1) if $(V_1, \rho_{V_1}) \not\cong (V_2, \rho_{V_2})$, then

$Hom_\Gamma(V_1, V_2) = \{0\}$

2) if $(V_1, \rho_{V_1}) \cong (V_2, \rho_{V_2})$
Let f be the isomorphism
$Hom_\Gamma(V_1, V_2) = \{\lambda f \mid \lambda \in \mathbb{C}\}$

Equivalently

$$\dim Hom_\Gamma(V_1, V_2) = \begin{cases} 0 & (V_1, \rho_{V_1}) \not\cong (V_2, \rho_{V_2}) \\ 1 & (V_1, \rho_{V_1}) \cong (V_2, \rho_{V_2}) \end{cases}$$

Proof

1) Let $f \in Hom_\Gamma(V_1, V_2)$ $f \neq 0$
Consider $\ker f$.
Claim $\ker f$ is Γ-invariant:
$x \in \ker f$ $g \in \Gamma$: $f(\rho_{V_1}(g)x) = \rho_{V_2}(g) \underbrace{f(x)}_{=0}$

$\Rightarrow \rho_{V_1}(g)x \in \ker f$.

$\Downarrow$

$\ker f = \{0\}$ or $\ker f = V_1$ because V_1 is irreducible.
$f \neq 0 \Rightarrow \ker f = \{0\} \Rightarrow f$ is injective
Similarly $Im f$ is Γ-invariant,
so $Im f = \{0\}$ or $Im f = V_2$
$f \neq 0 \Rightarrow Im f = V_2 \Rightarrow f$ is surjective
$\Rightarrow f$ is isomorphism: $(V_1, \rho_{V_1}) \cong (V_2, \rho_{V_2})$
contradiction.

2) $(V_1, \rho_{V_1}) \cong (V_2, \rho_{V_2})$, let us identify V_1 and V_2. $(V_2, \rho_{V_2}) = (V_1, \rho_{V_1})$.
Take $\varphi \in Hom_\Gamma(V_1, V_1)$
assume $\varphi \neq \lambda Id_{V_1}$ $(\lambda \in \mathbb{C})$.
Since $\varphi \in Hom_\Gamma(V_1, V_1)$

$\varphi \rho(g) = \rho(g) \varphi$ $\forall g \in \Gamma \Rightarrow$

if λ is an eigenvalue of φ,
let $E_\lambda = \{x \in V_1 : \varphi x = \lambda x\}$.
Then E_λ is Γ-invariant:
$\varphi x = \lambda x \Rightarrow \varphi(\rho(g)x) = \rho(g)\varphi(x) = \lambda \rho(g)x$. $\forall g \in \Gamma$
$\Rightarrow \rho(g)x \in E_\lambda$.
$E_\lambda \neq 0$
V_1 irreducible $\Rightarrow E_\lambda = V_1 \Rightarrow$
$\forall x \in V_1$ $\varphi(x) = \lambda x$. $\blacksquare$

Corollary

Suppose Γ is finite, (V, ρ_V) is a fin. dim Γ-rep.

By "decomposing" + Maschke

$$(V, \rho_V) = (V_1, \rho_{V_1}) \oplus \dots (V_n, \rho_{V_n})$$

So that (V_i, ρ_{V_i}) is irreducible $\forall i$

Let us group isomorphic reps together so that

$$(*)\ (V, \rho_V) = (V_1, \rho_{V_1})^{\oplus n_1} \oplus (V_2, \rho_{V_2})^{\oplus n_2} \oplus \dots \oplus (V_k, \rho_{V_k})^{\oplus n_k}$$

means: $\underbrace{(V_1, \rho_{V_1}) \oplus \dots \oplus (V_1, \rho_{V_1})}_{n_1}$

and $(V_i, \rho_{V_i}) \not\cong (V_j, \rho_{V_j})$ $(i \neq j)$.

Then n_i can be determined as $n_i = \dim \mathrm{Hom}_\Gamma(V_i, V)$.

(Hence such decomposition $(*)$ is unique for a given (V, ρ_V)).

Pf $\oplus$ commutes with Hom_Γ:

$$\mathrm{Hom}_\Gamma(U_1, U_2 \oplus U_3) = \mathrm{Hom}_\Gamma(U_1, U_2) \oplus \mathrm{Hom}(U_1, U_3)$$

$$\Rightarrow \mathrm{Hom}_\Gamma(V_i, V) =$$

$$= \mathrm{Hom}_\Gamma\left(V_i, V_1^{\oplus n_1} \oplus \dots \oplus V_k^{\oplus n_k}\right) =$$

$$= \mathrm{Hom}_\Gamma(V_i, V_1)^{\oplus n_1} \oplus \dots \oplus \mathrm{Hom}_\Gamma(V_i, V_k)^{\oplus n_k}$$

$$\left(\mathrm{Hom}_\Gamma(V_i, V_j) = 0 \quad j \neq i \quad (\text{Schur})\right)$$

$$= \underbrace{\mathrm{Hom}_\Gamma(V_i, V_i)}_{\cong\ \mathbb{C} \text{ by Schur}}{}^{\oplus n_i} = \mathbb{C}^{n_i}.$$

$\blacksquare$

Formula

Notation: (V_1, ρ_{V_1}) (V_2, ρ_{V_2}) f. dim reps of finite group Γ.

$\chi_{V_1}(g) := Tr\, \rho_{V_1}(g)$

$\chi_{V_2}(g) := Tr\, \rho_{V_2}(g)$. Then:

$$\dim Hom_\Gamma(V_1, V_2) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_{V_1}(g^{-1}) \chi_{V_2}(g).$$

Before the proof:

V_1, V_2 irreducible $\Rightarrow$

$$(\chi_{V_1}, \chi_{V_2}) = \begin{cases} 1 & (V_1, \rho_{V_1}) \cong (V_2, \rho_{V_2}) \\ 0 & \text{otherwise.} \end{cases}$$

where $(\ ,\ )$ is defined by

$$(\alpha, \beta) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \alpha(g^{-1}) \beta(g) \text{ for}$$

$\alpha, \beta : \Gamma \to \mathbb{C}$ are functions.

$\Rightarrow$ Orthogonality of characters.

<u>Pf</u> Plan:

1) Define an action of Γ on $Hom(V_1, V_2)$ so that $\{f \in Hom_\Gamma(V_1, V_2)$ if $\rho(g) f = f$ $\forall g \in \Gamma\}$

1) reduces the problem of computing $\dim Hom_\Gamma(V_1, V_2)$ to the problem of computing $\dim(W^\Gamma)$

$\{x \in W : \rho(g) x = x\}$

where W is a rep. of Γ

2) Compute $\dim(W^\Gamma)$ for arbitrary representation W.

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1), more details:

<u>Construction.</u> Given (V_1, ρ_{V_1}), (V_2, ρ_{V_2}) Γ-reps construct a new rep. $(Hom(V_1, V_2), \rho)$ by:

- $Hom(V_1, V_2)$ was defined earlier
- $\rho(g)(f) \in Hom(V_1, V_2) : x \mapsto \rho_{V_2}(g)\, f\, \rho_{V_1}(g)^{-1} x$

$g \in \Gamma$ $f \in Hom(V_1, V_2)$

again:

$$\rho(g)(f)(x) = \rho_{V_2}(g)(f(\rho_{V_1}(g)^{-1}(x))).$$

<u>Claim</u> $(Hom(V_1, V_2), \rho)$ is a rep of Γ of dimension $\dim V_1 \dim V_2$.

linear in f ✓

$\rho(e)(f)(x) = f(x)$ ✓

$\rho(g_1 g_2)(f)(x) = g_1 g_2 f (g_1 g_2)^{-1} x$

$\rho(g_1)(\rho(g_2)(f))(x) = g_1 (g_2 f g_2^{-1}) g_1^{-1} x.$

skip "ρ" ∎

<u>Notation</u> (W, ρ_W) is a Γ-rep, then $W^\Gamma := \{x \in W \mid \rho_W(g) x = x \ \forall g \in \Gamma\}$.

<u>Prop</u> $Hom(V_1, V_2)^\Gamma = Hom_\Gamma(V_1, V_2)$.

<u>Proof</u> $f \in Hom(V_1, V_2)$

$\rho(g) f = f \iff f \rho_{V_1}(g) = \rho_{V_2}(g) f$ ∎

Step 2) Given (W, ρ_W) a fin. dim Γ-rep we have

$$\dim(W^\Gamma) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} tr\, \rho_W(g).$$

<u>Pf</u> Let $\pi = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \rho_W(g)$.

$\pi : W \to W$.

Observations:

1) $x \in W^\Gamma \Rightarrow \pi(x) = x$

2) $\forall x \in W$: $\pi(x) \in W^\Gamma$:

<u>Pf</u> $\forall g \in \Gamma$

$$\rho_W(g) \pi(x) = \frac{1}{|\Gamma|} \sum_{g' \in \Gamma} \rho_W(g) \rho_W(g') x$$

$gg' = g''$

$$\pi(x) = \frac{1}{|\Gamma|} \sum_{g'' \in \Gamma} \rho_W(g'') x \quad ∎$$

3) $1) + 2) \Rightarrow \pi^2 = \pi$ $(\pi(x) \in W^\Gamma)$

4) $Im\, \pi = W^\Gamma$

The problem is reduced to computing $\dim Im\, \pi$ for a projector π.

<u>Trick</u> $\dim Im\, \pi = Tr(\pi)$

(in some basis

$$\pi = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & 0 \\ & & \ddots & & & \\ & & & 1 & & \\ & 0 & & & 0 & \\ & & & & & \ddots \ 0 \end{pmatrix} \quad \text{since } \pi^2 = \pi)$$

$Tr\, \pi = \#$ of 1 here. $= \dim Im\, \pi$.

Summary

$\dim Hom_\Gamma \longleftrightarrow \dim W^\Gamma$ $(W = Hom(V_1, V_2))$

$\dim W^\Gamma \longleftrightarrow Tr\, \pi$

$$Tr\, \pi = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} Tr\, \rho_W(g).$$