Algebraic Geometry WS20
Exercise set 4.

Instructor: Anton Mellit

Problem 1 (Eisenbud, p. 80, Ex. 2.6, but hint there doesn't help)**.** Let R be a ring, and let $I_1, \dots, I_n$ be ideals of R such that $I_i + I_j = R$ for all $i \neq j$. Show that $R/(\cap_i I_i) \cong \prod_i R/I_i$ by proving injectivity and surjectivity of the map $\varphi : R/(\cap_i I_i) \cong \prod_i R/I_i$ obtained from the n projection maps $R \to R/I_i$. Describe the ringed space associated to R in terms of the ringed spaces associated to R/I_i.

Problem 2 (Eisenbud, p. 79, Ex. 2.2, an alternate construction of localization)**.** Let R be a ring and let $U \subset R$ be any set. Let S be the multiplicative set generated by U (the set of all products of elements of U). Show that $S^{-1}R$ is isomorphic to the quotient of the polynomial ring $R[\{x_u\}_{u\in U}]$, with one variable for each element of U by, the ideal generated by $ux_u - 1$:

$$S^{-1}R = R[\{x_u\}_{u\in U}]/(\{ux_u - 1\}_{u\in U}).$$

Problem 3 (Eisenbud, p. 79, Ex. 2.3, how to localize without admitting it)**.** Suppose S is a multiplicatively closed subset of R. Show that there is a one-to-one correspondence, preserving sums and intersections, between ideals of $S^{-1}R$ and ideals I of R satisfying $(I : f) = I$ for all $f \in S$. Here $(I : f) = \{r \in R : fr \in I\}$. Show that for any ideal $I \subset R$ the ideal corresponding to $S^{-1}I$ is the ideal

$$\sum_{f\in S} (I : f^\infty), \quad \text{where} \quad (I : f^\infty) = \cup_{i=1}^{\infty} (I : f^i).$$

Historically, constructions like $(I : f)$ were used before localizations were defined, to accomplish the same ends.

Problem 4 (Eisenbud-Harris, p.20, Ex. I-20)**.** Describe the points, the topology, and the structure sheaf of each of the following schemes:

(1) $X_1 = \operatorname{Spec} \mathbb{C}[x]/(x^2)$,
(2) $X_2 = \operatorname{Spec} \mathbb{C}[x]/(x^2 - x)$,
(3) $X_3 = \operatorname{Spec} \mathbb{C}[x]/(x^3 - x^2)$,
(4) $X_4 = \operatorname{Spec} \mathbb{R}[x]/(x^2 + 1)$.

Due date: 6.11.2020, 9:45