Algebraic Geometry WS20
Exercise set 8.

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Problem 1 (Hartshorne, p. 82, Ex. 2.19)**.** Let R be a ring. Prove that the following are equivalent:

(1) $\operatorname{Spec}(R)$ is disconnected;
(2) there exist non-zero elements $e_1, e_2 \in R$ such that $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$ (these elements are called orthogonal idempotents);
(3) R is isomorphic to the product $R_1 \times R_2$ of two non-zero rings.

Problem 2 (Hartshorne, p. 79, Ex. 2.3 (Reduced schemes))**.** A scheme X is called *reduced* if for every open set $U \subset X$ the ring $O_X(U)$ has no non-zero nilpotent elements.

(1) Show that X is reduced if and only if for every point $x \in X$ the local ring $O_{X,x}$ has no non-zero nilpotent elements.
(2) Let X be a quasi-compact scheme. Let X_{red} be a ringed space constructed as follows. As a topological space X_{red} is simply X. For any open set $U \subset X$ let $O_{X_{\mathrm{red}}}(U)$ be the quotient of $O_X(U)$ by the nil-radical. Show that X_{red} is a scheme and construct a morphism $\iota : X_{\mathrm{red}} \to X$ which is the identity on the underlying topological space. The scheme X_{red} is called *the reduced scheme associated to* X.
(3) For any reduced scheme Y, show that any morphism $Y \to X$ can be uniquely decomposed as the composition $Y \to X_{\mathrm{red}} \xrightarrow{\iota} X$.

Problem 3 (From the last time)**.** Construct a surjective morphism $X \to \mathbb{P}^1$ where X is an affine scheme.

Due date: 04.12.2020, 9:45