Algebraic Geometry WS20
Exercise set 9.

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Problem 1 (Hartshorne, p. 79, Ex. 2.3 (Reduced schemes), from the last time)**.** A scheme X is called *reduced* if for every open set $U \subset X$ the ring $O_X(U)$ has no non-zero nilpotent elements.

(1) Show that X is reduced if and only if for every point $x \in X$ the local ring $O_{X,x}$ has no non-zero nilpotent elements.

(2) Let X be a quasi-compact scheme. Let X_{red} be a ringed space constructed as follows. As a topological space X_{red} is simply X. For any open set $U \subset X$ let $O_{X_{\mathrm{red}}}(U)$ be the quotient of $O_X(U)$ by the nil-radical. Show that X_{red} is a scheme and construct a morphism $\iota : X_{\mathrm{red}} \to X$ which is the identity on the underlying topological space. The scheme X_{red} is called *the reduced scheme associated to* X.

(3) For any reduced scheme Y, show that any morphism $Y \to X$ can be uniquely decomposed as the composition $Y \to X_{\mathrm{red}} \xrightarrow{\iota} X$.

Problem 2. Let R be a graded ring, and let $\mathfrak{p} \subset R$ be a prime ideal, not necessarily graded. Let $\mathfrak{q}$ be the ideal generated by the homogeneous elements of $\mathfrak{p}$. Show that $\mathfrak{q}$ is also prime.

Problem 3. Let $R = k[x, y]$ be the graded ring in which R_n is the span of monomials $x^i y^j$ with $2i + 3j = n$, i.e. the degree of x is 2 and the degree of y is 3. Describe $\mathrm{Proj}(R)$ as a set in the case

(1) if k is algebraically closed;

(2) if k is arbitrary.

Due date: 11.12.2020, 9:45