Algebraic Topology SS19
Exercise set 2.

Instructor: Anton Mellit

Problem 1. [Hatcher, Ch. 1.1, Ex. 3] For a path-connected space X, show that $\pi_1(X)$ is abelian if and only if all basepoint-change homomorphisms β_h depend only on the endpoints of the path h.

Problem* 2. [Hatcher, Theorem 1.10] Prove Borsuk-Ulam theorem in dimension two: For every continuous map $f : S^2 \to \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.

Problem 3. [Hatcher, Ch. 1.1, Ex. 15] Given a map $f : X \to Y$ and a path $h : I \to X$ from x_0 to x_1, show that $\pi_1(f)\beta_h = \beta_{fh}\pi_1(f)$ in the diagram below.

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\beta_h} & \pi_1(X, x_1) \\ \downarrow{\scriptstyle \pi_1(f)} & & \downarrow{\scriptstyle \pi_1(f)} \\ \pi_1(Y, f(x_0)) & \xrightarrow{\beta_{fh}} & \pi_1(Y, f(x_1)) \end{array}$$

Problem 4. [Hatcher, Ch. 0, Ex. 10] Show that a space X is contractible if and only if for every space Y every map $f : X \to Y$ is nullhomotopic (homotopic to a constant map). Similarly, show that X is contractible if and only if every map $f : Y \to X$ is nullhomotopic.

Problem 5. [Hatcher, Ch. 0, Ex. 11] Let $f : X \to Y$. Suppose there exists $g, h : Y \to X$ such that $f \circ g \cong \mathrm{Id}_Y$ and $h \circ f \cong \mathrm{Id}_X$. Show that f is a homotopy equivalence. More generally, prove the "2 out of 6 property":

$$Z \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{h} W$$

If $f \circ g$ and $h \circ f$ are homotopy equivalences, then f, g, h and $h \circ f \circ g$ are.

Due date: 26.03.2019