Algebraic Topology SS19
Exercise set 6.

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Problem 1. [Hatcher, p. 79, Ex. 8] (left from the last time) Let X and Y be path connected, locally path connected, semilocally simply connected spaces. Suppose X and Y are homotopy equivalent. Prove that the corresponding universal covering spaces $\tilde{X}$ and $\tilde{Y}$ are homotopy equivalent.

Problem 2. [Hatcher, p. 79, Ex. 5] (I found more problems with the solution presented last time) Let X be the subspace of $\mathbb{R}$ consisting of the four sides of the square $[0,1] \times [0,1]$ together with the segments $\{1/m\} \times [0,1]$ for $m = 2, 3, 4, \ldots$. Show that for any covering $\tilde{X} \to X$ there is a neighborhood of the left edge of X that lifts homeomorphically to $\tilde{X}$. Deduce that X has no universal covering space.

Problem 3. [Hatcher, Proposition 1.33] Let $\rho : \tilde{X} \to X$ be a covering and let $f : Y \to X$ be a continuous map. Suppose Y is path-connected and locally path-connected. Choose basepoints $y_0 \in Y$, $x_0 \in X$, $\tilde{x}_0 \in \tilde{X}$. such that $\rho(\tilde{x}_0) = f(y_0) = x_0$. Suppose $f_*(\pi_1(Y, y_0)) \subset \rho_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Show that there exists lifting $\tilde{f} : Y \to \tilde{X}$ with $\tilde{f}(y_0) = \tilde{x}_0$ and $\rho \circ \tilde{f} = f$.

Problem 4. Suppose X is a path connected, locally connected and locally simply connected space, let $x_0 \in X$. Let $\rho : \tilde{X} \to X$ be a covering. The automorphism group $\mathrm{Aut}(\rho)$ is the group of homeomorphisms $g : \tilde{X} \to \tilde{X}$ satisfying $\rho \circ g = \rho$. We call a covering ρ *normal* if $\mathrm{Aut}(\rho)$ acts transitively on $\rho^{-1}(x_0)$. Show that connected normal coverings of X are in bijection with normal subgroups of $\pi_1(X, x_0)$. Show that for a normal covering corresponding to a normal subgroup $H \subset \pi_1(X, x_0)$, the automorphism group is isomorphic to the quotient $\pi_1(X, x_0)/H$ (Hint: use functoriality from the lecture). A connected covering $\rho' : X' \to X$ is called a subcovering of $\rho : \tilde{X} \to X$ if there exists a map $f : \tilde{X} \to X'$ such that $\rho = \rho' \circ f$. Show that subcoverings of a normal covering $\rho : \tilde{X} \to X$ are in bijection with subgroups of $\mathrm{Aut}(\rho)$ (main theorem of Galois theory).

Problem 5. [Hatcher, p. 79, Ex. 9] let X be S^2 with 3 points removed. Construct a non-trivial covering of X with trivial automorphism group.

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