# Algebraic Topology SS19
# Exercise set 10.

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**Definition 0.1.** For a map $f : S^n \to S^n$ the *degree* is the number $d$ such that the homomorphism $f_* : H_n(S^n) \to H_n(S^n)$ is given by the multiplication by $d$

**Problem 1.** A map $S^n$ is called a *reflection* if it is the reflection with respect to a hyperplane passing through the origin. Show that any two reflections are homotopic, and show that the degree of a reflection is $-1$ (Hint: use the presentation of $S^n$ as a simplicial complex with two simplices).

**Problem 2.** What is the degree of a map $f : S^n \to S^n$ in the following two situations:

(1) $f$ is not surjective;
(2) $f$ has no fixed points?

**Problem 3.** [Hatcher, Ex. 7, p. 155] Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map. Show that the induced map $f_* : H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ is identity if $\det f > 0$ and $-\mathrm{Id}$ otherwise (Hint: show that $f$ is homotopic through invertible linear maps to the identity or to a reflection).

**Problem 4.** Let $X$ be a finite CW complex with $n_i$ cells of dimension $i$ for every $i$. Show that

$$\chi(X) = \sum_i (-1)^i n_i.$$

**Problem 5.** [Hatcher, Ex. 2, p. 155] Show that for every map $f : S^{2n} \to S^{2n}$ there is a point $x \in S^{2n}$ with $f(x) = x$ or $f(x) = -x$. Recall that the real projective space $\mathbb{R}P^k$ is constructed from $S^k$ by identifying $x$ with $-x$ for each $x \in S^k$. Deduce that every map $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point (Hint: what is the universal cover of $\mathbb{R}P^{2n}$?) Construct a map $f : \mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without fixed points from a linear transformation $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors.

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