Algebraic Topology SS19
Exercise set 11.

Instructor: Anton Mellit

Problem 1 (Hatcher, p. 53, Ex. 11)**.** The *mapping torus* T_f of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with the corresponding point $(f(x), 1)$. In the case $X = S^1 \vee S^1$ (wedge of circles) with f basepoint-preserving compute a presentation of $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$. (Hint: One way to do this is to regard T_f as built from $X \vee S^1$ by attaching cells).

Problem 2 (Hatcher, p. 54, Ex. 14)**.** Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one 3-cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$, of order eight.

Problem 3 (Hatcher, p. 79, Ex. 10)**.** Find all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$, up to isomorphism of covering spaces without basepoints.

Problem 4 (Hatcher, p. 80, Ex. 14)**.** Find all the connected covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

Problem 5 (Hatcher, p. 131, Ex. 8)**.** Construct a 3-dimensional simplicial complex X from n tetrahedra $T_1, \dots, T_n$ by the following two steps. First arrange the tetrahedra in a cyclic pattern (a figure is given in the book), so that each T_i shares a common vertical face with T_{i+1}, subscripts being taken modulo n. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show the simplicial homology groups of X in dimensions 0, 1, 2, 3 are $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, 0, $\mathbb{Z}$. (The space X is an example of a *lens space*.)

Problem 6. We call a simplicial complex *good* if there is no pair of simplices with same set of vertices. Show that the second barycentric subdivision of a simplicial complex is good and give an example when the first barycentric subdivision is not good.

Problem 7 (Hatcher, p. 133, Ex. 28)**.** Let X be the cone on the 1-skeleton of Δ^3, the union of all line segments joining points in the six edges of Δ^3 to the barycenter of Δ^3. Compute the local homology groups $H_n(X, X \setminus \{x\})$ for all $x \in X$. Define ∂X to be the subspace of points x such that $H_n(X, X \setminus \{x\}) = 0$ for all n, and compute the local homology groups $H_n(\partial X, \partial X \setminus \{x\})$. Use these calculations to determine which subsets $A \subset X$ have the property that $f(A) \subset A$ for all homeomorphisms $f : X \to X$.

Problem 8 (Hatcher, p. 133, Ex. 31)**.** In the notations of the five-lemma

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, "c"] & D \arrow[r, "j"] \arrow[d, "d"] & E \arrow[d, "e"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "j'"] & E' \end{tikzcd}$$

give an example where the maps a, b, d, e are zero, but c is nonzero.

Problem 9 (Hatcher, p. 156, Ex. 12)**.** Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show via covering spaces that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.

Problem 10 (Hatcher, p. 157, Ex. 28)**.**

(1) Use the Mayer–Vietoris sequence to compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus.

(2) Do the same for the space obtained by attaching a Möbius band to $\mathbb{R}P^2$ via a homeomorphism of its boundary circle to the standard $\mathbb{R}P^1 \subset \mathbb{R}P^2$.

Due date: 24.06.2019, 12:00