# Commutative algebra WS18
# Exercise set 11.

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**Problem 1.** [AM, Ch. 3, Ex. 11, continuation] Let $R$ be a ring in which every prime ideal is maximal. Show that for every prime ideal $\mathfrak{p} \subset R$ the localization $(R/\operatorname{Nil}(R))_{\mathfrak{p}}$ is a field. Deduce that $R/\operatorname{Nil}(R)$ is absolutely flat.

**Problem 2.** Let $R'$ be a flat $R$-ring via a homomorphism $\varphi : R \to R'$. Suppose for every non-zero $R$-module $M$, we have $R' \otimes_R M \neq 0$ ($R'$ is faithfully flat). Show that

(1) A sequence of $R$-modules $M_1 \to M_2 \to M_3$ is exact if an only if the corresponding sequence $R' \otimes_R M_1 \to R' \otimes_R M_2 \to R' \otimes_R M_3$ is exact.
(2) For every $R$-module $M$, the natural mapping $M \to R' \otimes_R M$ is injective.
(3) For all ideals $I \subset R$, we have $\varphi^{-1}(R'\varphi(I)) = I$.
(4) If $f : R'' \to R$ is such that $\varphi \circ f$ is flat, then $f$ itself is flat.

**Problem 3.** [AM, Ch. 4, Ex. 4] Show that in the polynomial ring $\mathbb{Z}[t]$, the ideal $\mathfrak{m} = (2, t)$ is maximal, the ideal $(4, t)$ is $\mathfrak{m}$-primary, but not a power of $\mathfrak{m}$.

**Problem 4.** [AM, Lemma 4.4 iii)] Let $\mathfrak{q}$ be a primary ideal of $R$, $\mathfrak{p} = \operatorname{rad}(\mathfrak{q})$, and let $x \in R \setminus \mathfrak{p}$. Show that $(\mathfrak{q} : x) = \mathfrak{q}$.

**Problem 5.** [AM, Ch. 4, Ex. 12] Let $R$ be a ring, $S$ a multiplicatively closed subset of $R$. For any ideal $\mathfrak{a}$, let $S(\mathfrak{a})$ denote the contraction of $S^{-1}\mathfrak{a}$ in $R$ (the preimage of $S^{-1}\mathfrak{a}$ with respect to the map $R \to S^{-1}R$). The ideal $S(\mathfrak{a})$ is called the *saturation* of $\mathfrak{a}$ with respect to $S$. Prove that

(1) $S(\mathfrak{a}) \cap S(\mathfrak{b}) = S(\mathfrak{a} \cap \mathfrak{b})$
(2) $S(\operatorname{rad}(\mathfrak{a})) = \operatorname{rad}(S(\mathfrak{a}))$
(3) $S(\mathfrak{a}) = R \Leftrightarrow \mathfrak{a}$ meets $S$
(4) $S_1(S_2(\mathfrak{a})) = (S_1S_2)(\mathfrak{a})$.

If $\mathfrak{a}$ has a primary decomposition, prove that the set of ideals $S(\mathfrak{a})$ where $S$ runs through the set of all multiplicatively closed subsets of $R$ is finite.

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