# Commutative algebra WS18
# Exercise set 12.

**Instructor: Anton Mellit**

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**Problem 1.** [AM, Ch. 4, Ex. 8] Show that in the polynomial ring $k[x_1, \dots, x_n]$ the ideals $(x_1, \dots, x_l)$ $(1 \le l \le n)$ are prime and all their powers are primary.

**Problem 2.** [AM, Ch. 4, Ex. 10] For any prime ideal $\mathfrak{p}$ in a ring $R$, and for any ideal $\mathfrak{a}$, denote by $S_\mathfrak{p}(\mathfrak{a})$ the saturation with respect to $S_\mathfrak{p} = R \setminus \mathfrak{p}$. Show that it is given by the kernel of the map $R \to (R/\mathfrak{a})_\mathfrak{p}$. Prove that

(1) $S_\mathfrak{p}(\mathfrak{a}) \subset \mathfrak{p}$;
(2) $\operatorname{rad}(S_\mathfrak{p}(\mathfrak{a})) = \mathfrak{p}$ if and only if $\mathfrak{p}$ is a minimal associated prime of $\mathfrak{a}$;
(3) if $\mathfrak{p} \supset \mathfrak{p}'$, then $S_\mathfrak{p}(\mathfrak{a}) \subset S_{\mathfrak{p}'}(\mathfrak{a})$;
(4) $\bigcap_{\mathfrak{p} \in D(\mathfrak{a})} = \mathfrak{a}$, where $D(\mathfrak{a})$ is the set of associated primes of $\mathfrak{a}$.

**Problem 3.** [AM, Ch. 4, Ex. 11] If $\mathfrak{p}$ is a minimal prime ideal of a ring $R$, show that $S_\mathfrak{p}(0)$ is the smallest $\mathfrak{p}$-primary ideal. Let $\mathfrak{a}$ be the intersection of all the ideals $S_\mathfrak{p}(0)$ as $\mathfrak{p}$ runs through the minimal prime ideals of $R$. Show that $\mathfrak{a}$ is contained in the nilradical of $R$. Suppose the zero ideal has a primary decomposition. Prove that $\mathfrak{a} = 0$ if and only if every prime ideal associated to 0 is minimal.

**Problem 4.** [AM, Ch. 4, Ex. 13] Let $R$ be a ring and $\mathfrak{p}$ a prime ideal of $R$. The *$n$-th symbolic power of* $\mathfrak{p}$ is defined to be the saturation

$$\mathfrak{p}^{(n)} = S_\mathfrak{p}(\mathfrak{p}^n).$$

Show that

(1) $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal;
(2) if $\mathfrak{p}^n$ has a primary decomposition, then $\mathfrak{p}^{(n)}$ is its $\mathfrak{p}$-primary component;
(3) if $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$ has a primary decomposition, then $\mathfrak{p}^{(m+n)}$ is its $\mathfrak{p}$-primary component.
(4) $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ holds if and only if $\mathfrak{p}^n$ is $\mathfrak{p}$-primary.

*Due date: 22.01.2019*