Commutative algebra WS18
Exercise set 13.

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Problem 1. Let $\mathfrak{a} \subset R$ be an ideal, and let $\mathfrak{p}$ be a minimal prime containing $\mathfrak{a}$. Suppose $\mathfrak{a}$ has a primary decomposition and let $\mathfrak{q}$ be its $\mathfrak{p}$-primary component. Show that $\mathfrak{q} = S_{\mathfrak{p}}(\mathfrak{a})$. Deduce that the $\mathfrak{p}$-primary component for a minimal prime $\mathfrak{p}$ does not depend on the choice of decomposition (Corollary 4.11). Hint: use properties of the saturation proved in earlier exercises.

Problem 2. [AM, Ch. 4, Ex. 11] Let $\mathfrak{a}$ be the intersection of all the ideals $S_{\mathfrak{p}}(0)$ as $\mathfrak{p}$ runs through the minimal prime ideals of R. Show that $\mathfrak{a}$ is contained in the nilradical of R. Suppose the zero ideal has a primary decomposition. Prove that $\mathfrak{a} = 0$ if and only if every prime ideal associated to 0 is minimal. Hint: use Problem 1.

Problem 3. Let $R = k[x, a, b]$ and let $J = (x^2 - a, x^3 - b)$. Find a Groebner basis for R using Buchberger algorithm with respect to the deglex monomial order. Verify that $a^3 - b^2 \in J$ using the found basis.

Problem 4. With the same R and J as in Problem 3, find a Groebner basis with respect to the lex order. Use the obtained information to determine $J \cap k[a, b]$.

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