Commutative algebra WS18
Exercise set 2.

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Problem 1. [AM Ch. 1, Ex. 7] Let R be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in R is maximal.

Problem 2. [AM Ch. 1, Ex. 8] Let R be a non-zero ring. Show that the set of prime ideals of R has minimal elements with respect to inclusion.

Problem 3. [AM Ch. 1, Ex. 10] Let R be a ring. Show that the following are equivalent:

(1) R has exactly one prime ideal;
(2) every element of R is either unit or nilpotent.

Problem 4. [AM, Ch. 1, Ex. 11] A ring R is *Boolean* if $x^2 = x$ for all $x \in R$. In a Boolean ring R, show that

(1) $2x = 0$ for all $x \in R$;
(2) every prime ideal $\mathfrak{p}$ is maximal and $R/\mathfrak{p}$ is the field with two elements;
(3) every finitely generated ideal is principal.

Problem 5. [AM, Ch. 1, Ex. 12] Suppose a ring R has exactly one maximal ideal (such rings are called *local*). Show that if $x \in R$ satisfies $x^2 = x$ (such element is called *idempotent* or *projector*), then $x = 0$ or $x = 1$.

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