Commutative algebra WS18
Exercise set 3.

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Problem 1. Let R be a ring. Show that R is an integral domain if and only the following conditions are satisfied:

(1) R has exactly one minimal prime ideal;
(2) every nilpotent element in R is zero.

Problem 2. Let R be a ring. A *derivation* on R is a map $d : R \to R$ satisfying

(1) $d(f + g) = d(f) + d(g)$,
(2) $d(fg) = fd(g) + gd(f)$

for all $f, g \in R$. Construct a bijection between the set of all derivations on R and a subset of the set of ring homomorphisms $R \to R[x]/(x^2)$.

Problem 3. Let $R = k[x, y, z]$ for a field k. Let $\mathfrak{a} = (y, z)$, $\mathfrak{b} = (y - x^2, z)$. Compute $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a}\mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$. Find the dimension of the quotients $R/(\mathfrak{a} + \mathfrak{b})$, $\mathfrak{a} \cap \mathfrak{b}/\mathfrak{a}\mathfrak{b}$ as vector spaces over k.

Problem 4. The *Jacobson radical* of R is defined as the intersection of all maximal ideals of R. Show that $x \in R$ belongs to the Jacobson radical if and only if $1 + xy$ is a unit for all $y \in R$.

Problem 5. Let $\mathfrak{a}, \mathfrak{b}$ be ideals in a ring R. Suppose $\mathfrak{p}$ is a prime ideal such that $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$. Show that

(1) $\mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p}$;
(2) $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.

Show that in the following statements (1) implies (2), and (2) implies (3):

(1) $\mathfrak{a}\mathfrak{b} = \mathfrak{p}$;
(2) $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{p}$;
(3) $\mathfrak{a} = \mathfrak{p}$ or $\mathfrak{b} = \mathfrak{p}$.

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