Commutative algebra WS18
Exercise set 5.

Instructor: Anton Mellit

Problem 1. [AM, Ch. 3, Ex. 5](Leftover from set 4) Let R be a ring. Suppose that, for each prime $\mathfrak{p}$, $R_{\mathfrak{p}}$ has no nilpotent elements $\neq 0$. Show that R has no nilpotent elements $\neq 0$. If each $R_{\mathfrak{p}}$ is an integral domain, is R necessarily an integral domain?

Problem 2. [AM, Proposition 3.11 (v)] Let S be a multiplicatively closed subset of a ring R. For any ideal $\mathfrak{a} \subseteq R$, denote by $S^{-1}\mathfrak{a} \subseteq S^{-1}R$ the ideal consisting of elements a/s for $a \in \mathfrak{a}$, $s \in S$. Show that the operation S^{-1} commutes with formation of finite sums, products, intersections and radicals.

Problem 3. Let S be a multiplicatively closed subset of a ring R with canonical homomorphism $\pi : R \to S^{-1}R$, and let $\mathfrak{p} \subset S^{-1}R$ be a prime ideal. Show that $S^{-1}\pi^*\mathfrak{p} = \mathfrak{p}$. Is it true that $S^{-1}\mathfrak{p}$ is prime for any prime $\mathfrak{p} \subset R$?

Problem 4. Let S be a multiplicatively closed subset of a ring R, and let $\mathfrak{a} \subseteq R$ be an ideal. Denote by $S + \mathfrak{a}$ the corresponding subset of $R/\mathfrak{a}$. Show that the ring $(S + \mathfrak{a})^{-1}(R/\mathfrak{a})$ is isomorphic to the ring $S^{-1}R/S^{-1}\mathfrak{a}$.

Problem 5. Let R be a ring and let $\mathfrak{p} \subset R$ be a prime ideal. Let $\mathfrak{m}$ be the maximal ideal of $R_{\mathfrak{p}}$. Show that the field of fractions of $R/\mathfrak{p}$ is isomorphic to $R_{\mathfrak{p}}/\mathfrak{m}$.

Due date: 13.11.2018