Commutative algebra WS18
Exercise set 7.

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Problem 1. [AM, Ch. 2, Ex. 1] Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if and only if m and n are coprime.

Problem 2. [AM, Ch. 2, Ex. 3] Let R be a local ring, M and N finitely generated R-modules. Prove that if $M \otimes_R N = 0$, then $M = 0$ or $N = 0$.

Problem 3. [AM, Ch. 2, Ex. 5+6] For any R-module M, let $M[x]$ be the set of all polynomials in x with coefficients in M, that is expressions of the form

$$m_0 + m_1 x + \cdots m_r x^r \qquad (m_i \in M).$$

Defining product in the usual way, show that $M[x]$ is an $R[x]$-module. Show that

$$M[x] \cong R[x] \otimes_R M.$$

Show that $R[x]$ is flat as a module over R.

Problem 4. [AM, Ch. 2, Ex. 8] Show that

(1) If M and N are flat R-modules, then so is $M \otimes_R N$.
(2) Let $\varphi : R \to R'$ be a homomorphism of rings (in such a case we say that R' is an R-algebra, or a ring over R). Let M be a flat R'-module. Suppose R' is flat (as an R-module). Show that M viewed as an R-module is also flat.

Problem 5. [AM, Ch. 2, Ex. 10] Let R be a ring, and let $I \subset R$ be an ideal contained in the Jacobson radical of R. Let M be an R-module and let N be a finitely generated R-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/IM \to N/IN$ is surjective, show that u is surjective. Does injectivity of $M/IM \to N/IN$ imply injectivity of u?

Due date: 27.11.2018