Commutative algebra WS18
Exercise set 8.

Instructor: Anton Mellit

Problem 1. [AM, Ch. 2, Ex. 4] Let $(M_i)_{i\in I}$ be a family of R-modules, and let M be their direct sum. Prove that M is flat if and only if each M_i is flat.

Problem 2. Let $R = \mathbb{C}[x, y]$, $R' = \mathbb{C}[x, y, z]/(x - yz)$. View R' as an R-algebra via the map $R \to R'$ which sends x to x and y to y (the corresponding map of spaces is called the affine blow-up). Is R' flat over R?

Problem 3. Find a ring representing the functor F in the following situations. Below we describe only the set $F(R')$ for any ring R', and the map $F(\varphi) : F(R') \to F(R'')$ for any ring homomorphism $\varphi : R' \to R''$ is clear.

(1) For each ring R' we have $F(R')$ is the set consisting of one element.
(2) For each ring R' we have $F(R') = R'$.
(3) Ring R is fixed, and for each ring R' we have $F(R') = \mathrm{Hom}(R, R') \times R'$ (cartesian product).

Problem 4. [AM, Ch. 2, Ex. 10] (left from the last time) Let R be a ring, and let $I \subset R$ be an ideal contained in the Jacobson radical of R. Let M be an R-module and let N be a finitely generated R-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/IM \to N/IN$ is surjective, show that u is surjective. Does injectivity of $M/IM \to N/IN$ imply injectivity of u?

Problem 5. [AM, Ch. 2, Ex. 13] Let $R \to R'$ be a ring homomorphism, and let M be an R'-module. Form an R'-module $M' = R' \otimes_R M_R$, where M_R is M viewed as an R-module. Show that the homomorphism $g : M \to M'$ defined by $g(x) = 1 \otimes x$ is injective, and show that $g(M)$ is a direct summand of M'.

Due date: 4.12.2018