Commutative algebra WS18
Exercise set 9.

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Problem 1. [AM, Ch. 2, Ex. 10] (left from the last to last time) Let R be a ring, and let $I \subset R$ be an ideal contained in the Jacobson radical of R. Let M be an R-module and let N be a finitely generated R-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/IM \to N/IN$ is surjective, show that u is surjective. Does injectivity of $M/IM \to N/IN$ imply injectivity of u?

Problem 2. (from last time) Find a ring representing the functor F in the following situations. Below we describe only the set $F(R')$ for any ring R', and the map $F(\varphi) : F(R') \to F(R'')$ for any ring homomorphism $\varphi : R' \to R''$ is clear.

(1) For each ring R' we have $F(R')$ is the set consisting of one element.
(2) For each ring R' we have $F(R') = R'$.
(3) Ring R is fixed, and for each ring R' we have $F(R') = \mathrm{Hom}(R, R') \times R'$ (cartesian product).

Problem 3. Suppose R is a ring and

$$0 \to M \to N \to P \to 0$$

a short exact sequence of R-modules. Is it true that for any R-module Q the sequence

$$0 \to \mathrm{Hom}(P, Q) \to \mathrm{Hom}(N, Q) \to \mathrm{Hom}(M, Q) \to 0$$

is exact? The same question about

$$0 \to \mathrm{Hom}(Q, M) \to \mathrm{Hom}(Q, N) \to \mathrm{Hom}(Q, P) \to 0.$$

Problem 4 (part of AM, Ch. 2, Ex. 27)**.** Suppose R is a ring such that every module is flat (such rings are called *absolutely flat.*) Show that every principal ideal is generated by an idempotent ($x \in R$ such that $x^2 = x$). Show that every finitely generated ideal is principal and therefore a direct summand of R. Show that every non-unit is a zero divisor.

Problem 5. Let $R = \mathbb{C}[x, y]$ and let $J \subset R$ be the ideal (x, y). Compute the kernel of the multiplication map $J \otimes_R J \to J$.

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