Last time:

Cohomology of $\mathbb{C}P^n = \mathbb{Q}[x] / x^{n+1}=0$

$x \in H^2(\mathbb{C}P^n; \mathbb{Q})$.

Computed that the number of solutions to a system of equations of degrees $d_1, \dots, d_n$ is $d_1 \dots d_n$.

Missing points: *) If $Z_1, Z_2 \subset X$ intersect transversally, then $[Z_1] \cup [Z_2] = [Z_1 \cap Z_2]$

$\boxed{\begin{array}{c} \text{cup} = \cup \\ \text{cap} = \cap \end{array}}$

||

$Z_1 \cap Z_2$ is a complex manifold and locally around $Z_1 \cap Z_2$, Z_1 and Z_2 look like transversally intersecting vector spaces.

<u>Today</u> K-theory, Chern classes, Hirzebruch-Riemann-Roch.

HRR

dimensions of vector spaces $\longleftrightarrow$ Cohomology (like last time)

dim(space of homogeneous polynomials of degree d in $n+1$ variables) $\quad \mathbb{P}^n \quad$ $\mathbb{Q}[x]/x^{n+1}$

<u>Q</u>: Compute dim.

Cases:

$n=1$ (2 variables x, y)

d	
1	x, y
2	x^2, xy, y^2
3	x^3, x^2y, xy^2, y^3
d	$d+1 = \binom{d+1}{1}$

3 variables x, y, z

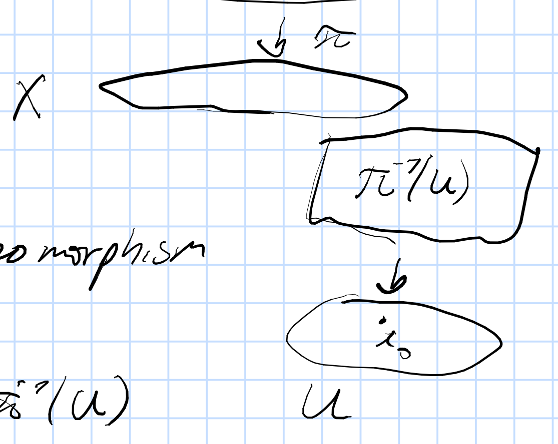
1	x, y, z	3
2	$x^2, xy, y^2, xz, yz, z^2$	6
3	10	
d	$\binom{d+2}{2}$	

general n : $\binom{d+n}{n}$.

Ingredients

<u>K theory</u> Studying <u>vector bundles</u> on manifolds.

<u>Def</u> <u>vector bundle</u> on a topological space X is a topological space $\mathcal{E}$ together with a projection map $\pi: \mathcal{E} \to X$, each fiber $\pi^{-1}(x) = \mathcal{E}_x$ has a structure of a vector space, $\forall x_0 \in X$ $\exists\, U \ni x_0$ an open neighborhood and a homeomorphism



$h: U \times \mathbb{R}^n \xrightarrow{\sim} \pi^{-1}(U)$

so that

1)
$$\begin{tikzcd} U \times \mathbb{R}^n \arrow[r, "h"] \arrow[d, "\pi_1"] & \pi^{-1}(U) \arrow[d, "\pi"] \\ U \arrow[r, "Id_U"] & U \end{tikzcd}$$
is commutative

2) $\forall x \in U$ $\quad h|_{\pi_1^{-1}(x)} : \mathbb{R}^n \to \pi^{-1}(x)$ is a vector space isomorphism.

h are called local trivializations.
If n is the same for all x, n is called the rank of $\mathcal{E}$.

<u>Examples</u> 1) trivial bundle

$$\mathcal{E} = X \times \mathbb{R}^n \qquad \pi = \pi_1$$

2) tangent bundle (X is a manifold)

Say $X \subset \mathbb{R}^m$ $\quad m$ large enough

$$\mathcal{E} \subset \mathbb{R}^{2m} = \left\{ (x, v) \,\middle|\, \begin{matrix} x \in X \\ v \in \mathbb{R}^m \end{matrix} \,\middle|\, v \in T_x X \right\}$$

$\pi: \mathcal{E} \to X \qquad \pi = \pi_1$

$\qquad (x, v) \to x$

How to build an interesting algebraic theory out of vector bundles?

<u>Def</u> Fix X.

Consider formal linear combinations of vector bundles on X modulo relations:
$\forall$ $\mathcal{E}_1, \mathcal{E}_2$ vector bundles

$$(*) \qquad [\mathcal{E}_1 \oplus \mathcal{E}_2] = [\mathcal{E}_1] + [\mathcal{E}_2]$$

elements look like $\quad n_1[\mathcal{E}_1] + n_2[\mathcal{E}_2] + \dots n_k[\mathcal{E}_k]$

<u>Observation</u> any element can be written as a difference $[\mathcal{E}] - [F]$.

The resulting abelian group is called the Grothendieck K-group $K(X)$.

<u>Def</u> $\mathcal{E}_1 \oplus \mathcal{E}_2$:

$$\text{this is } \left\{ v_1 \in \mathcal{E}_1, v_2 \in \mathcal{E}_2 \,\middle|\, \pi(v_1) = \pi(v_2) \right\}.$$

$$\downarrow \pi(v_1) = \pi(v_2)$$

$$X$$

$$\operatorname{rank}(\mathcal{E}_1 \oplus \mathcal{E}_2) = \operatorname{rank}(\mathcal{E}_1) \oplus \operatorname{rank}(\mathcal{E}_2).$$

<u>Example</u> $\quad X \subset \mathbb{R}^m \qquad TX$ tangent bundle

$\qquad \forall x \in X \longrightarrow T_x X$

$$(T_x X)^{\perp} = N_x X$$

$\pi: NX \to X$ is the normal bundle.

$$TX \oplus NX = X \times \mathbb{R}^m \quad \text{(trivial bundle)}$$

the trivial bundle of rank $m = \underbrace{X \times \mathbb{R} \oplus \dots X \times \mathbb{R}}_{m \text{ copies}} \Rightarrow$

$$[TX] + [NX] = m\,[X \times \mathbb{R}]$$

Usualy we denote the class of $[X \times \mathbb{R}]$ by 1.

So we get $\quad [TX] + [NX] = m.$

Characteristic classes

Stiefel-Whitney classes:

To a bundle $\mathcal{E}$ of rank n on X we associate $\quad w_1(\mathcal{E}), w_2(\mathcal{E}) \dots w_n(\mathcal{E})$

$$w_1(\mathcal{E}) \in H^1(X, \mathbb{Z}/2\mathbb{Z}) \qquad w_n(\mathcal{E}) \in H^n(X, \mathbb{Z}/2\mathbb{Z}).$$

This is usualy written as

$$w(\mathcal{E}) = 1 + w_1(\mathcal{E}) + w_2(\mathcal{E}) \dots + w_n(\mathcal{E}) \in H^0(X, \mathbb{Z}/2) \oplus H^1(X, \mathbb{Z}/2) \oplus \dots$$

$w(\mathcal{E})$ is called the total Stiefel-Whitney class.

We don't need them.

S-W classes satisfy $\quad w(\mathcal{E}_1 \oplus \mathcal{E}_2) = w(\mathcal{E}_1)\, w(\mathcal{E}_2)$

$$\Updownarrow$$

$$w_k(\mathcal{E}_1 \oplus \mathcal{E}_2) = \sum_{i=0}^{k} w_i(\mathcal{E}_1)\, w_{k-i}(\mathcal{E}_2).$$

In enumerative geometry we need to replace real manifolds by complex manifolds
real vector bundles by complex vector bundles
S-W classes by Chern classes.

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1) X complex manifold, e.g. $\mathbb{C}P^n$.

around every point $x_0 \in X$ we have a chart

$$U \ni x_0$$

$$h: D^n \to U$$

$$D = \{z \in \mathbb{C} \mid |z| < 1\}$$

change of coordinates maps are holomorphic.
(for spaces arising in algebraic geometry they are often given by rational functions).

2) Complex vector bundles:

$\pi: \mathcal{E} \to X$ $\quad \forall x \in X \quad \mathcal{E}_x = \pi^{-1}(x)$ is a complex vector space

projection $\quad$ total space

local triviality $\quad h: U \times \mathbb{C}^n \to \pi^{-1}(U)$.
homeomorphism.

Holomorphic vector bundles: h is holomorphic.
on complex manifolds

3) Grothendieck K-group of complex vector bundles in the same way as real!

But 1 difference for holomorphic vector bundles.

We replace the relation $[\mathcal{E}_1 \oplus \mathcal{E}_2] = [\mathcal{E}_1] + [\mathcal{E}_2]$

by:

For any short exact sequence of holomorphic vector bundles

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

we have $[\mathcal{E}] = [\mathcal{E}_1] + [\mathcal{E}_2]$.

$\mathcal{E}_1 \to \mathcal{E}$ $\quad$ for each $x \in X$ we have a linear map $\mathcal{E}_{1x} \to \mathcal{E}_x$ and it is injective.
(+ $\mathcal{E}_1 \to \mathcal{E}$ is holomorphic)

$\mathcal{E} \to \mathcal{E}_2$ $\quad$ the same, but surjective

and $\operatorname{Im}(\mathcal{E}_{1x} \to \mathcal{E}_x) = \ker(\mathcal{E}_x \to \mathcal{E}_{2x})$.

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4) Chern classes $c_i(\mathcal{E}) \in H^{2i}(X, \mathbb{Z})$

+ Similar properties as $w_i(\mathcal{E})$.
