I hope everyone can read this

Last time:

Motivation, example with 3 projectors

Definition of representation of a group

looked at the case of group $\mathbb{Z}$

Isomorphism of reps

Direct sums, indecomposable reps.

Plan of the course

1) Finite groups
2) Lie algebras
3) Finite dimensional algebras

Today: find a first interesting representation of a finite group

Let us eliminate „not interesting" groups (abelian ‖ commutative)

Proposition. If Γ is finite abelian, then any indecomposable finite dimensional representation is 0 or 1 dimensional.

Pf Let (V, ρ_V) be an indecomposable representation of Γ, $\dim V \geq 2$

Idea: find an element $g \in \Gamma$ such that $\rho_V(g)$ has distinct eigenvalues.

Observation For any $g \in \Gamma$ $\rho_V(g)$ is diagonalizable.

Pf Consider the Jordan form of $\rho_V(g)$. Take a Jordan block $J = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$.

Since Γ is finite, $g^n = e$ (some $n > 0$), hence $\rho_V(g)^n = Id_V$. $J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & & \\ & \lambda^n & & \\ & & \ddots & \\ 0 & & & \lambda^n \end{pmatrix} \neq Id$ $\quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}$

$\Rightarrow \rho_V(g)^n \neq Id_V$ contradiction

Case 1 For all g we have $\rho_V(g)$ has the same eigenvalues, i.e.

by observation $\rho_V(g) = \lambda_g \cdot Id$ $\quad \lambda_g \in \mathbb{C}$.

Let us write $V = U_1 \oplus U_2$ in an arbitrary way with $U_1 \subset V$ $U_2 \subset V$, $U_1 \neq 0$, $U_2 \neq 0$. Then $\forall g \in \Gamma$ $\rho_V(g) U_1 \subset U_1$, $\rho_V(g) U_2 \subset U_2$, so U_1, U_2 are representations of Γ and $V \cong U_1 \oplus U_2$ as a representation, contradiction.

Case 2 There is $g \in \Gamma$ such that $\rho_V(g)$ has distinct eigenvalues. $\rho_V(g)$ in some basis looks like

$$\begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_1 & & & & \\ & & \lambda_2 & & & \\ & & & \lambda_2 & & \\ & & & & \ddots & \\ & & & & & \lambda_m \end{pmatrix}$$

$\lambda_i \neq \lambda_j$ $(i \neq j)$

Let $V_1 = \{x \in V \mid \rho_V(g) x = \lambda_1 x\}$

$V_2 = \sum_{i=2}^{m} \{x \in V \mid \rho_V(g) x = \lambda_i x\}$

Clearly $\underbrace{V = V_1 \oplus V_2}_{\text{means that } V = V_1 + V_2,\ V_1 \cap V_2 = \{0\}}$. We only need to prove that $\forall g' \in \Gamma$ we have $\rho_V(g') V_1 \subset V_1$, $\rho_V(g') V_2 \subset V_2$, then V_1, V_2 are representations of Γ and $V \cong V_1 \oplus V_2$.

To prove this: if x is s.t. $\rho_V(g) x = \lambda_i x$

then $\rho_V(g) \rho_V(g') x = \rho_V(gg') x \underset{\text{by commutativity}}{=} \rho_V(g'g) x = \rho_V(g') \rho_V(g) x$
$= \lambda_i \rho_V(g') x$

Therefore $\rho_V(g')$ preserves the eigenspace decomposition, so we are done.

Now: abelian groups are easy.

For example, Let $\Gamma = \mathbb{Z}/n\mathbb{Z}$ $(= \mathbb{Z}_n)$

All indecomposables are 1-dimensional

$\rho_V(g)$ is 1×1 matrix, so is just a number

ρ_V is determined by $\rho_V(1) = \lambda$,

$1 = \rho_V(0) = \rho_V(\underbrace{1 + \dots + 1}_{n}) = \lambda^n \Rightarrow \lambda = e^{\frac{2\pi i}{n} k}$

$k \in \{0, \dots, n-1\}$

So we have n representations, corresponding to choices of k.

Having abelian groups dealt with, we consider non-abelian.

The smallest non-abelian group is S_3

it has $3! = 6$ elements. How to construct representations?

Construction If group Γ acts on a finite set S

(homomorphism $\varphi : \Gamma \to S_m$ some m)

take $V = \mathbb{C}^m$ with basis $e_1, \dots, e_m$

Let $\rho_V(g)\, e_i = e_{\varphi(g)(i)}$, extend to V by linearity.
$g \in \Gamma$

This is a representation called permutation representation

$$\left(\begin{array}{ccc} \rho_V(g_1 g_2) e_i & & \rho_V(g_1)\rho_V(g_2) e_i \\ \| & & \| \\ \underbrace{e_{\varphi(g_1 g_2)(i)}} & & \underbrace{e_{\varphi(g_1)(\varphi(g_2)(i))}} \\ & \overset{\shortparallel}{(\varphi(g_1)\,\varphi(g_2))(i)} \overset{\shortparallel}{} & \end{array} \right).$$

Using this idea, we have a 3-dimensional representation

$V = \mathbb{C}^3$, basis e_1, e_2, e_3

$g \in S_3$: $\rho_V(g)\, e_i = e_{g(i)}$

Notice : $e_1 + e_2 + e_3$ is invariant vector

also the orthogonal complement $(e_1 + e_2 + e_3)^\perp$ is an invariant subspace

alternatively $U = \{ x_1 e_1 + x_2 e_2 + x_3 e_3 \,/\, (x_1, x_2, x_3) \in \mathbb{C}^3,\ x_1 + x_2 + x_3 = 0 \}$

Pf $\ulcorner$ $\rho_V(g)(e_1 + e_2 + e_3) = \sum_{i=1}^{3} e_{g(i)} = \sum_{i=1}^{3} e_i$.

if $x_1 e_1 + x_2 e_2 + x_3 e_3$ satisfies $x_1 + x_2 + x_3 = 0$, then

$\rho_V(g)\left(\sum_{i=1}^{3} x_i e_i \right) = \sum_{i=1}^{3} x_i e_{g(i)} = \sum_{i=1}^{3} x_{g^{-1}(i)} e_i$ also

satisfies $\sum_{i=1}^{3} x_{g^{-1}(i)} = \sum_{i=1}^{3} x_i = 0$. $\lrcorner$

In fact

let $U' = \mathbb{C}(e_1 + e_2 + e_3)$, then clearly

$\mathbb{C}^3 = U \oplus U'$. $\left(x_1 e_1 + x_2 e_2 + x_3 e_3 = \underbrace{\frac{x_1 + x_2 + x_3}{3}(e_1 + e_2 + e_3)}_{U'} + \underbrace{(\text{elem. of } U)}_{U} \right)$

$\dim U' = 1$

$\dim U = 2$

Claim U is indecomposable.

Pf Otherwise $U = U_1 \oplus U_2$, where $\dim U_i = 1$.

Pick $0 \neq V = x_1 e_1 + x_2 e_2 + x_3 e_3 \in U_1$.

$\tau_{23} = \begin{pmatrix} 1 \to 1 \\ 2 \to 3 \\ 3 \to 2 \end{pmatrix}$ (more generally $\tau_{ij} = \begin{pmatrix} 1 \to 1 \\ \vdots \\ i \to j \\ j \to i \end{pmatrix}$)

Case 1 $x_1 \neq 0$.

$\tau_{23} V = (x_1, x_3, x_2)$ must be proportional to V.

$x_1 \neq 0 \Rightarrow$ equal to V

$\Rightarrow x_2 = x_3 = c$, $x_1 = -2c$, $V = (-2c, c, c)$ $(c \neq 0)$

$\tau_{12} V$ is not proportional to V $\cancel{\ }$

Case 2 $x_1 = 0$ $V = (0, c, -c)$ $(c \neq 0)$

$\tau_{12} V$ is not proportional to V.

$\cancel{\ }$. $\lrcorner$

Remarks 1) Any permutation representation decomposes as a direct sum $U \oplus U'$ as for S_3.

$\sum_{i=1}^{m} x_i = 0$ $\qquad \mathbb{C}(e_1 + \dots + e_m)$

U is of course not always indecomposable.
(i.e. consider abelian group)

2) We can try to replace $\mathbb{C}$ by the field of residues modulo p ($p = 2, 3, 5, 7, \dots$ prime), but then it is not even true that for permutation reps we have $U \oplus U'$.

Take $p = 3$ $e_1 + e_2 + e_3$ satisfies

$\sum_{i=1}^{3} x_i = 3 = 0 \pmod 3$.

Many open interesting problems in this direction.
(modular rep. theory)

3) A good general strategy to construct representations is by taking permutation representations, decompose them into indecomposables.

Main theorem of rep. theory of finite groups:

All f. dim representations are classified by their trace functions a.k.a. characters.

if (V, ρ_V) is a rep. of Γ, then

character is the map $\chi_V : \Gamma \to \mathbb{C}$ given by

$\chi_V(g) = \operatorname{tr} \rho_V(g)$. (isomorphic reps produce the same trace functions)

Change basis by a matrix f:

$\rho_V(g) \to f \rho_V(g) f^{-1}$, $\operatorname{tr}(f \rho_V(g) f^{-1}) = \operatorname{tr}(\rho_V(g))$.

Our rep from before

Next time; its trace function.